Section 1.4: The Matrix Equation $A\mathbf{x} = \mathbf{b}$

- This section is about solving the "matrix equation" $A\mathbf{x} = \mathbf{b}$, where A is an $m \times n$ matrix and $\mathbf{b}$ is a column vector with m entries (both given in the question), and $\mathbf{x}$ is an unknown column vector with n entries (which we are trying to solve for). The first thing to know is what $A\mathbf{x}$ means: it means we are multiplying the matrix A times the vector $\mathbf{x}$. How do we multiply a matrix by a vector? We use the "row times column" rule, see the bottom of page 38 for examples.

- Solving $A\mathbf{x} = \mathbf{b}$ is the same as solving the system described by the augmented matrix $[A|\mathbf{b}]$.

- $A\mathbf{x} = \mathbf{b}$ has a solution if and only if $\mathbf{b}$ is a linear combination of the columns of A.

- Theorem 4 is very important, it tells us that the following statements are either all true or all false, for any $m \times n$ matrix A:

  (a) For every $\mathbf{b}$, the equation $A\mathbf{x} = \mathbf{b}$ has a solution.

  (b) Every column vector $\mathbf{b}$ (with m entries) is a linear combination of the columns of A.

  (c) The columns of A span $\mathbb{R}^m$ (this is just a restatement of (b), once you know what the word "span" means).

  (d) A has a pivot in every row.

  This theorem is useful because it means that if we want to know if $A\mathbf{x} = \mathbf{b}$ has a solution for every $\mathbf{b}$, we just need to check if A has a pivot in every row. **Note:** If A does not have a pivot in every row, that does *not* mean that $A\mathbf{x} = \mathbf{b}$ does not have a solution for some given vector $\mathbf{b}$. It just means that there are *some* vectors $\mathbf{b}$ for which $A\mathbf{x} = \mathbf{b}$ does not have a solution.

- Finally, it is very useful to know that multiplying a vector by a vector has the following nice properties:

  (a) $A(\mathbf{u} + \mathbf{v}) = A(\mathbf{u}) + A(\mathbf{v})$, for vectors $\mathbf{u}, \mathbf{v}$

  (b) $A(c\mathbf{u}) = cA(\mathbf{u})$, for vectors $\mathbf{u}$ and scalars c.

Section 1.5: Solution Sets of Linear Systems

- A **homogeneous** system is one that can be written in the form $A\mathbf{x} = \mathbf{0}$. Equivalently, a homogeneous system is any system $A\mathbf{x} = \mathbf{b}$ where $\mathbf{x} = 0$ is a solution (notice that this means that $\mathbf{b} = 0$, so both definitions match). The solution $\mathbf{x} = 0$ is called the **trivial solution**. A solution $\mathbf{x}$ is **non-trivial** is $\mathbf{x} \neq \mathbf{0}$.

- The homogeneous system $A\mathbf{x} = \mathbf{0}$ has a non-trivial solution if and only if the equation has at least one free variable (or equivalently, if and only if A has a column with no pivots).

- **Parametric vector form:** Let's say you have found the solution set to a system, and the free variables are x_3, x_4, x_5. Then to write the solution set in 'parametric vector form' means to write the solution as

$$\mathbf{x} = \mathbf{p} + x_3\mathbf{u} + x_4\mathbf{v} + x_5\mathbf{w}$$

where $\mathbf{p}, \mathbf{u}, \mathbf{v}, \mathbf{w}$ are vectors with numerical entries. A method for writing a solution set in this form is given on page 46.

Section 1.7: Linear Independence

- Like everything else in linear algebra, the definition of **linear independence** can be phrased in many different equivalent ways. $\mathbf{v}_1, \mathbf{v}_2, \cdots, \mathbf{v}_p$ are linearly independent if any of the following equivalent statements are true:
  (a) the vector equation $x_1\mathbf{v}_1 + x_2\mathbf{v}_2 + \cdots + x_2\mathbf{v}_2 = 0$ has only the trivial solution
  (b) none of the vectors $\mathbf{v}_1, \mathbf{v}_2, \cdots, \mathbf{v}_p$ are a linear combination of the others
  (c) if we put the vectors together as columns of the matrix A, then the system $A\mathbf{x} = \mathbf{0}$ has only the trivial solution
  (c) if we put the vectors together as columns of the matrix A, then the system $A\mathbf{x} = \mathbf{0}$ has only the trivial solution
  (d) if we put the vectors together as columns of the matrix A, then A has a pivot in every column
- If vectors aren't linearly independent, then they are **linearly dependent**. This means that (at least) one of the vectors is a linear combination of the rest. **Note:** This does not mean that all of the vectors are linear combinations of the others. See the following exercise.
- **Exercise 1:** Find three vectors in $\mathbb{R}^3$ that are linearly dependent, but where the third vector is not a linear combination of the first two.
- **Method to check linear (in)dependence:** If we want to check if a set of given vectors is linearly independent, put them together as columns of a matrix, and then row reduce the matrix. If there is a pivot in every column, then they are independent. Otherwise, they are dependent.
- **Exercise 2 (1.7.1):** Check if the following vectors are linearly independent:

$$\begin{bmatrix} 5 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 7 \\ 2 \\ -6 \end{bmatrix}, \begin{bmatrix} 9 \\ 4 \\ -8 \end{bmatrix}$$

- Theorem 9: Any set containing the zero vector is linearly dependent. This follows immediately from the method above, because if one of the columns is zero, there can't be a pivot in every column (there are other easy ways to prove this theorem also, see the book for example).
- Theorem 8: If we have p vectors, each with n entries, and $p > n$, then these vectors have to be linearly dependent. (This follows from the method above too, because if there are more columns than rows, there can't be a pivot in every column).
- **Exercise 3:** Find 2 vectors in $\mathbb{R}^5$ that are linearly dependent. Notice that this means that if $p \leq n$ in the theorem above, then the vectors might be dependent *or* independent.