Classification of certain types of maximal matrix subalgebras

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Abstract

Let $\mathcal{M}_n(K)$ denote the algebra of $n \times n$ matrices over a field K of characteristic zero. A nonunital subalgebra $\mathcal{N} \subset \mathcal{M}_n(K)$ will be called a *nonunital intersection* if $\mathcal{N}$ is the intersection of two unital subalgebras of $\mathcal{M}_n(K)$. Appealing to recent work of Agore, we show that for $n \geq 3$, the dimension (over K) of a nonunital intersection is at most $(n-1)(n-2)$, and we completely classify the nonunital intersections of maximum dimension $(n-1)(n-2)$. We also classify the unital subalgebras of maximum dimension properly contained in a parabolic subalgebra of maximum dimension in $\mathcal{M}_n(K)$.

1 Introduction

Let $\mathcal{M}_n(F)$ denote the algebra of $n \times n$ matrices over a field F. For some interesting sets Λ of subspaces $\mathcal{S} \subset \mathcal{M}_n(F)$, those $\mathcal{S} \in \Lambda$ of maximum dimension over F have been completely classified. For example, a theorem of Gerstenhaber and Serezhkin [7, Theorem 1] states that when Λ is the set of subspaces $\mathcal{S} \subset \mathcal{M}_n(F)$ for which every matrix in $\mathcal{S}$ is nilpotent, then each $\mathcal{S} \in \Lambda$ of maximum dimension is conjugate to the algebra of all strictly upper triangular matrices in $\mathcal{M}_n(F)$. For another example, it is shown in [1, Prop. 2.5] that when Λ is the set of proper unital subalgebras $\mathcal{S} \subset \mathcal{M}_n(F)$ and F is an algebraically closed field of characteristic zero, then each $\mathcal{S} \in \Lambda$ of maximum dimension is a parabolic subalgebra of maximum dimension in $\mathcal{M}_n(F)$.

The goal of this paper is to classify the elements in Λ of maximum dimension in the cases $\Lambda = \Gamma$ and $\Lambda = \Omega$, where the sets Γ and Ω are defined below. First we need some definitions.

Write $\mathcal{M} = \mathcal{M}_n = \mathcal{M}_n(K)$, where K is a field of characteristic zero. (It would be interesting to know if this restriction on the characteristic can be relaxed for the results in this paper.) In the spirit of [3, p. viii], we define a subalgebra of $\mathcal{M}$ to be a vector subspace of $\mathcal{M}$ over K closed under the multiplication of $\mathcal{M}$ (cf. [3, p. 2]); thus a subalgebra need not have a unity, and the unity of a unital subalgebra need not be a unity of the parent algebra. Subalgebras $\mathcal{A}, \mathcal{B} \subset \mathcal{M}$ are said to be similar if $\mathcal{A} = \{S^{-1}BS : B \in \mathcal{B}\}$ for some invertible $S \in \mathcal{M}$.

In Isaac's text [4, p. 161], every ring is required to have a unity, but the unity in a subring need not be the same as the unity in its parent ring. Under this definition, a ring may have subrings whose intersection is not a subring. This motivated us to study examples of pairs of unital subalgebras in $\mathcal{M}$ whose intersection $\mathcal{N}$ is nonunital. We call such $\mathcal{N}$ a *nonunital intersection* and we let Γ denote the set of all nonunital intersections $\mathcal{N} \subset \mathcal{M}$. Note that Γ is closed under transposition and conjugation, i.e., if $\mathcal{N} \in \Gamma$, then $\mathcal{N}^{\mathrm{T}} \in \Gamma$ and $S^{-1}\mathcal{N}S \in \Gamma$ for any invertible $S \in \mathcal{M}$.

In order to define Ω, we need to establish additional notation. Let $\mathcal{M}[R_n]$ denote the subalgebra of $\mathcal{M}$ consisting of those matrices whose n-th row is zero. Similarly, $\mathcal{M}[R_n, C_n]$ indicates that the n-th row and n-th column are zero, etc. For $1 \leq i, j \leq n$, let $E_{i,j}$ denote the elementary matrix in $\mathcal{M}$ with a single entry 1 in row i, column j, and 0 in each of the other $n^2 - 1$ positions. The identity matrix in $\mathcal{M}$ will be denoted by I. For the maximal parabolic subalgebra $\mathcal{P} := \mathcal{M}[R_n] + KE_{n,n}$ in $\mathcal{M}$, define Ω to be the set of proper subalgebras $\mathcal{B}$ of $\mathcal{P}$ with $\mathcal{B} \neq \mathcal{M}[R_n]$.

We now describe Theorems 3.1–3.3, our main results. Theorem 3.1 shows that $\dim \mathcal{N} \leq (n-1)(n-2)$ for each $\mathcal{N} \in \Gamma$. Theorem 3.2 shows that up to similarity, $\mathcal{W} := \mathcal{M}[R_n, R_{n-1}, C_n]$ and $\mathcal{W}^{\mathrm{T}} := \mathcal{M}[R_n, C_{n-1}, C_n]$ are the only subalgebras in Γ having maximum dimension $(n-1)(n-2)$. In Theorem 3.3, we show that $\dim \mathcal{B} \leq n^2 - 2n + 3$ for each $\mathcal{B} \in \Omega$, and we classify all $\mathcal{B} \in \Omega$ of maximum dimension $n^2 - 2n + 3$.

The proofs of our theorems depend on four lemmas, which are proved in Section 2. Lemma 2.1 shows that $\mathcal{W}$ (and hence also $\mathcal{W}^{\mathrm{T}}$) is a nonunital intersection of dimension $(n-1)(n-2)$ when $n \geq 3$. Lemmas 2.2 and 2.3 show that $\dim \mathcal{L} \leq n(n-1)$ for any nonunital subalgebra $\mathcal{L} \subset \mathcal{M}$, and when equality holds, $\mathcal{L}$ must be similar to $\mathcal{M}[R_n]$ or $\mathcal{M}[C_n]$. (Thus if Λ denotes the set of nonunital subalgebras $\mathcal{L} \subset \mathcal{M}$, Lemmas 2.2 and 2.3 classify those $\mathcal{L} \in \Lambda$ of maximum dimension.) Lemma 2.4 shows that if $\mathcal{U} \subset \mathcal{M}$ is a subalgebra with unity different from I, then some conjugate of $\mathcal{U}$ is contained in $\mathcal{M}[R_n, C_n]$.

2 Lemmas

Recall the definition $\mathcal{W} := \mathcal{M}[R_n, R_{n-1}, C_n]$.

Lemma 2.1. *For $n \geq 3$, $\mathcal{W} \in \Gamma$ and* $\dim \mathcal{W} = (n-1)(n-2)$.

Proof. For $n > 1$, define $A \in \mathcal{M}$ by $A = I + E_{n,n-1}$. Note that $A^{-1} = I - E_{n,n-1}$. A computation shows that for $M \in \mathcal{M}[R_n, C_n]$, the conjugate

AMA^{-1} is obtained from M by replacing the (zero) bottom row of M by the $(n-1)$-th row of M. Since the bottom two rows of AMA^{-1} are identical, it follows that

$$AMA^{-1} \in \mathcal{M}[R_n, C_n] \cap A\mathcal{M}[R_n, C_n]A^{-1} \text{ if and only if } AMA^{-1} \in \mathcal{W}.$$

Since $\mathcal{W} = A^{-1}\mathcal{W}A$, this shows that $\mathcal{W}$ is the intersection of the unital subalgebras $A^{-1}\mathcal{M}[R_n, C_n]A$ and $\mathcal{M}[R_n, C_n]$. To see that $\mathcal{W}$ is nonunital, note that $E_{1,n-1}$ is a nonzero matrix in $\mathcal{W}$ for which $E_{1,n-1}W$ is the zero matrix for each $W \in \mathcal{W}$; thus $\mathcal{W}$ cannot have a right identity, so $\mathcal{W} \in \Gamma$. Finally, it follows from the definition of $\mathcal{W}$ that $\dim \mathcal{W} = (n-1)(n-2)$. □

Remark: The same proof shows that $\mathcal{W} \in \Gamma$ when the field K is replaced by an arbitrary ring R with $1 \neq 0$. If R happens to be commutative, then the dimension of the algebra $\mathcal{W}$ over R is well defined [8, p. 483] and it equals $(n-1)(n-2)$.

Lemma 2.2. *For any nonunital subalgebra $\mathcal{L} \subset \mathcal{M}$, $\dim \mathcal{L} \leq n(n-1)$.*

Proof. If $\mathcal{L} + KI = \mathcal{M}$, then $\mathcal{L}$ would be a two-sided proper ideal of $\mathcal{M}$, contradicting the fact that $\mathcal{M}$ is a simple ring [8, p. 280]. Fergus Gaines [2, Lemma 4] proved that for *any* field F, the $F-$dimension of a proper unitary subalgebra of $\mathcal{M}_n(F)$ is at most $n^2 - n + 1$. (Agore [1, Cor. 2.6] proved this only for fields of characteristic zero.) Since $\mathcal{L} + KI$ is a proper subalgebra of $\mathcal{M}$ containing the unity I, it follows that

$$\dim \mathcal{L} = -1 + \dim (\mathcal{L} + KI) \leq n(n-1).$$

□

Lemma 2.3. *Any nonunital subalgebra $\mathcal{L} \subset \mathcal{M}$ with $\dim \mathcal{L} = n(n-1)$ must be similar to either $\mathcal{M}[R_n]$ or $\mathcal{M}[C_n] = \mathcal{M}[R_n]^{\mathrm{T}}$.*

Proof. Consider the two parabolic subalgebras $\mathcal{P}, \mathcal{P}' \subset \mathcal{M}$ defined by

$$\mathcal{P} = \mathcal{P}_K = \mathcal{M}[R_n] + KE_{n,n} , \quad \mathcal{P}' = \mathcal{P}'_K = \mathcal{M}[C_1] + KE_{1,1} .$$

Note that $\mathcal{P}'$ is similar to the transpose $\mathcal{P}^{\mathrm{T}}$. Since $\mathcal{L} + KI$ is a proper subalgebra of $\mathcal{M}$ of dimension $n(n-1)+1$, it follows from Agore [1, Prop. 2.5] that $\mathcal{L} + KI$ is similar to $\mathcal{P}$ or $\mathcal{P}'$, under the condition that K is algebraically closed. However, Nolan Wallach [9] has proved that this condition can be dropped; see the Appendix. Thus, replacing $\mathcal{L}$ by a conjugate if necessary, we may assume that $\mathcal{L} + KI = \mathcal{P}$ or $\mathcal{L} + KI = \mathcal{P}^{\mathrm{T}}$. We will assume that $\mathcal{L} + KI = \mathcal{P}$, since the proof for $\mathcal{P}^{\mathrm{T}}$ is essentially the same. It suffices to show that $\mathcal{L}$ is similar to $\mathcal{M}[R_n]$ or $\mathcal{M}[C_1]$, since $\mathcal{M}[C_1]$ is similar to $\mathcal{M}[C_n]$.

Assume temporarily that each $L \in \mathcal{L}$ has all entries 0 in its upper left $(n-1)\times(n-1)$ corner. Then $n = 2$, because if $n \geq 3$, then every matrix in $\mathcal{P}$ would have a zero entry in row 1, column 2, contradicting the definition of $\mathcal{P}$. Since $\mathcal{L} \subset \mathcal{M}_2[C_1]$ and both sides have dimension 2, we have $\mathcal{L} = \mathcal{M}_2[C_1]$, which proves the theorem under our temporary assumption.

When the temporary assumption is false, there exists $L \in \mathcal{L}$ with the entry 1 in row i, column j for some fixed pair i, j with $1 \leq i, j \leq n-1$. Since $E_{i,i}$ and $E_{j,j}$ are in $\mathcal{P} = \mathcal{L} + KI$ and $\mathcal{L}$ is a two-sided ideal of $\mathcal{P}$, we have $E_{i,j} = E_{i,i}LE_{j,j} \in \mathcal{L}$. Consequently, $E_{a,b} = E_{a,i}E_{i,j}E_{j,b} \in \mathcal{L}$ for all pairs a, b with $1 \leq a \leq n-1$ and $1 \leq b \leq n$. Therefore

$$\mathcal{M}[R_n] = \sum_{a=1}^{n-1}\sum_{b=1}^{n} KE_{a,b} \subset \mathcal{L},$$

and since both $\mathcal{M}[R_n]$ and $\mathcal{L}$ have the same dimension $n(n-1)$, we conclude that $\mathcal{L} = \mathcal{M}[R_n]$. □

Remark: Any subalgebra $\mathcal{B} \subset \mathcal{M}$ properly containing $\mathcal{M}[R_n]$ must also contain I. To see this, note that $\mathcal{B}$ contains a nonzero matrix of the form

$$B := \sum_{i=1}^{n} c_i E_{n,i}\ , \quad c_i \in K.$$

If $c_j = 0$ for all $j < n$, then $E_{n,n} \in \mathcal{B}$, so $I \in \mathcal{B}$. On the other hand, if $c_j \neq 0$ for some $j < n$, then $E_{n,n} = c_j^{-1}BE_{j,n} \in \mathcal{B}$, so again $I \in \mathcal{B}$.

Lemma 2.4. *Suppose that a subalgebra $\mathcal{U} \subset \mathcal{M}$ has a unity $e \neq I$. Then $S^{-1}\mathcal{U}S \subset \mathcal{M}[R_n, C_n]$ for some invertible $S \in \mathcal{M}$.*

Proof. Let r be the rank of the matrix e. Note that e is idempotent, so by [6, p. 27], there exists an invertible $S \in \mathcal{M}$ for which $S^{-1}eS = D_r$, where D_r is a diagonal matrix with entries 1 in rows 1 through r, and entries 0 elsewhere. Replacing $\mathcal{U}$ by $S^{-1}\mathcal{U}S$ if necessary, we may assume that $e = D_r$. Since $r \leq n-1$, we have

$$\mathcal{U} = e\mathcal{U}e \subset e\mathcal{M}e = D_r\mathcal{M}D_r \subset D_{n-1}\mathcal{M}D_{n-1} = \mathcal{M}[R_n, C_n].$$

□

3 Theorems

Recall that Γ is the set of all nonunital intersections in $\mathcal{M}$.

Theorem 3.1. *If $\mathcal{N} \in \Gamma$, then* $\dim \mathcal{N} \leq (n-1)(n-2)$.

Proof. Let $\mathcal{N} \in \Gamma$, so that $\mathcal{N} = \mathcal{U} \cap \mathcal{V}$ for some pair of unital subalgebras $\mathcal{U}, \mathcal{V} \subset \mathcal{M}$. Since $\mathcal{N}$ is nonunital, one of $\mathcal{U}, \mathcal{V}$, say $\mathcal{U}$, does not contain I. Thus $\mathcal{U}$ contains a unity $e \neq I$. Define S as in Lemma 2.4. Replacing $\mathcal{U}$, $\mathcal{V}$, $\mathcal{N}$ by $S^{-1}\mathcal{U}S$, $S^{-1}\mathcal{V}S$, $S^{-1}\mathcal{N}S$, if necessary, we deduce from Lemma 2.4 that $\mathcal{U}$ is contained in $\mathcal{M}[R_n, C_n]$. Since $\mathcal{N}$ is a nonunital subalgebra of $\mathcal{U} \subset \mathcal{M}[R_n, C_n]$, it follows from Lemma 2.2 with $(n-1)$ in place of n that $\dim \mathcal{N} \leq (n-1)(n-2)$. □

Theorem 3.2. *Let $n \geq 3$. Then up to similarity, $\mathcal{W}$ and $\mathcal{W}^{\mathrm{T}}$ are the only subalgebras of $\mathcal{M}$ in Γ having dimension $(n-1)(n-2)$.*

Proof. By Lemma 2.1, every subalgebra of $\mathcal{M}$ similar to $\mathcal{W}$ or $\mathcal{W}^{\mathrm{T}}$ lies in Γ and has dimension $(n-1)(n-2)$. Conversely, let $\mathcal{N} \in \Gamma$ with $\dim \mathcal{N} = (n-1)(n-2)$. We must show that $\mathcal{N}$ is similar to $\mathcal{W}$ or $\mathcal{W}^{\mathrm{T}}$.

We may assume, as in the proof of Theorem 3.1, that $\mathcal{N}$ is a nonunital subalgebra of $\mathcal{M}[R_n, C_n]$. Let $\mathcal{L}$ be the subalgebra of $\mathcal{M}_{n-1}$ consisting of those matrices in the upper left $(n-1) \times (n-1)$ corners of the matrices in $\mathcal{N}$. Since $\dim \mathcal{L} = \dim \mathcal{N} = (n-1)(n-2)$, it follows from Lemma 2.3 that $\mathcal{L}$ is similar to $\mathcal{M}_{n-1}[R_{n-1}]$ or $\mathcal{M}_{n-1}[C_{n-1}]$. Thus $\mathcal{N}$ is similar to $\mathcal{W} = \mathcal{M}[R_n, R_{n-1}, C_n]$ or $\mathcal{W}^{\mathrm{T}} = \mathcal{M}[R_n, C_{n-1}, C_n]$. □

Recall that Ω denotes the set of proper subalgebras $\mathcal{B} \neq \mathcal{M}[R_n]$ in $\mathcal{P}$.

Theorem 3.3. *Let $\mathcal{B} \in \Omega$. Then* $\dim \mathcal{B} \leq n^2 - 2n + 3$. *If $\mathcal{B}$ has maximum dimension $n^2 - 2n + 3$, then $\mathcal{B}$ is similar to one of*

$$\mathcal{M}E_{n,n} + \mathcal{M}[R_n, C_1] + KE_{1,1}, \qquad \mathcal{M}E_{n,n} + \mathcal{M}[R_n, R_{n-1}] + KE_{n-1,n-1}.$$

Proof. Let $e \in \mathcal{M}$ denote the diagonal matrix of rank $n-1$ with entry 0 in row n and entries 1 in the remaining rows. Because e is a left identity in $\mathcal{M}[R_n]$ and $\mathcal{B}e \subset \mathcal{M}[R_n, C_n]$, it follows that $\mathcal{B}e$ is an algebra.

First suppose that $\mathcal{B}e = \mathcal{M}[R_n, C_n]$. Then $\mathcal{P} = \mathcal{C} + \mathcal{D}$, where

$$\mathcal{C} = \mathcal{B} + KE_{n,n}, \quad \mathcal{D} = \mathcal{M}[R_n]E_{n,n}.$$

We proceed to show that $\mathcal{C} \cap \mathcal{D}$ is zero. Assume for the purpose of contradiction that there exists a nonzero matrix $B \in \mathcal{C} \cap \mathcal{D}$. Then $B \in \mathcal{B}$. We have $\mathcal{B}B = \mathcal{D}$, since the matrices in $\mathcal{B}$ have all possible submatrices in their upper left $(n-1)$ by $(n-1)$ corners. Thus $\mathcal{D} \subset \mathcal{B} \subset \mathcal{C}$, which implies that $\mathcal{M}[R_n] \subset \mathcal{B}$ and $\mathcal{P} = \mathcal{C} = \mathcal{B} + KE_{n,n}$. If $KE_{n,n} \subset \mathcal{B}$, then $\mathcal{B} = \mathcal{P}$, and if $KE_{n,n}$ is not contained in $\mathcal{B}$, then $\mathcal{B} = \mathcal{M}[R_n]$; either case contradicts the fact that $\mathcal{B} \in \Omega$.

Since $\mathcal{C} \cap \mathcal{D}$ is zero,

$$\dim \mathcal{B} \leq \dim \mathcal{C} = \dim \mathcal{P} - \dim \mathcal{D} = (n^2 - n + 1) - (n-1) = n^2 - 2n + 2.$$

Thus

$$\dim \mathcal{B} < n^2 - 2n + 3,$$

so the desired upper bound for $\dim \mathcal{B}$ holds when $\mathcal{B}e = \mathcal{M}[R_n, C_n]$.

Next suppose that $\mathcal{B}e$ is a proper subalgebra of $\mathcal{M}[R_n, C_n]$. We proceed to show that

$$d := \dim \mathcal{B}e \leq (n-1)(n-2) + 1,$$

by showing that

$$\dim \mathcal{L} \leq (n-1)(n-2) + 1$$

for every proper subalgebra $\mathcal{L}$ of $\mathcal{M}_{n-1}$. If $\mathcal{L}$ is nonunital, then

$$\dim \mathcal{L} \leq (n-1)(n-2) < (n-1)(n-2) + 1$$

by Lemma 2.2 (with $n-1$ in place of n). If $\mathcal{L}$ contains a unit different from the identity of $\mathcal{M}_{n-1}$, then by Lemma 2.4 (with $\mathcal{L}$ in place of $\mathcal{U}$),

$$\dim \mathcal{L} \leq \dim \mathcal{M}[R_{n-1}, C_{n-1}] = (n-2)^2 < (n-1)(n-2) + 1.$$

If $\mathcal{L}$ contains the identity of $\mathcal{M}_{n-1}$, then by [2, Lemma 4],

$$\dim \mathcal{L} \leq (n-1)(n-2) + 1.$$

This completes the demonstration that $d \leq (n-1)(n-2) + 1$.

Let $B_1 e, B_2 e, \ldots, B_d e$ be a basis for $\mathcal{B}e$, with $B_i \in \mathcal{B}$. Since $\mathcal{B}$ is a subspace of the vector space spanned by the $d + n$ matrices

$$B_1, \ldots, B_d, E_{1,n}, \ldots, E_{n,n},$$

it follows that

$$\dim \mathcal{B} \le d + n \le (n-1)(n-2) + 1 + n = n^2 - 2n + 3.$$

Thus the desired upper bound for dim $\mathcal{B}$ holds in all cases.

The argument above shows that when we have the equality

$$\dim \mathcal{B} = d + n = (n-1)(n-2) + 1 + n = n^2 - 2n + 3,$$

then

$$\mathcal{B} = \mathcal{B}e + \mathcal{M}E_{n,n}.$$

Moreover, from the equality $d = \dim \mathcal{B}e = (n-1)(n-2)+1$, it follows from Agore [1, Prop. 2.5] (again appealing to the Appendix to dispense with the condition of algebraic closure) that there is an invertible matrix S in the set $\mathcal{M}[R_n, C_n] + E_{n,n}$ such that $S^{-1}\mathcal{B}eS$ is equal to one of

$$\mathcal{M}[R_n, C_n, C_1] + KE_{1,1}, \qquad \mathcal{M}[R_n, C_n, R_{n-1}] + KE_{n-1,n-1}.$$

Since $S^{-1}\mathcal{M}E_{n,n}S = \mathcal{M}E_{n,n}$, we achieve the desired classification of Ω. $\square$

4 Appendix

Let F be a field of characteristic 0 with algebraic closure $\overline{F}$. Given a proper subalgebra $\mathcal{C} \subset \mathcal{M}_n(F)$ of maximum dimension, Agore [1, Prop. 2.5, Cor. 2.6] proved that the $\bar{F}$-span of $\mathcal{C}$ is similar over $\bar{F}$ to the $\bar{F}$-span of $\mathcal{D}$ for some parabolic subalgebra $\mathcal{D}$ of maximum dimension in $\mathcal{M}_n(F)$. The purpose of this Appendix is to deduce that $\mathcal{C}$ is similar over F to $\mathcal{D}$.

Lemma 4.1. (Wallach) *Let $\mathcal{A}$ be a subspace of $\mathcal{M}_n(F)$ of dimension $n-1$ such that $\mathcal{A} \otimes_F \overline{F}$ has basis of one of the following two forms:*

a) $x_1 \otimes \lambda_1, x_2 \otimes \lambda_1, ..., x_{n-1} \otimes \lambda_1$, with $\lambda_1 \in (\bar{F}^n)^*, x_j \in \bar{F}^n$ and $\lambda_1(x_j) = 0$,

b) $x_1 \otimes \lambda_1, x_1 \otimes \lambda_2, ..., x_1 \otimes \lambda_{n-1}$, with $\lambda_j \in (\bar{F}^n)^*, x_1 \in \bar{F}^n$ and $\lambda_j(x_1) = 0$.

Then in case a) $\mathcal{A}$ is $F-$conjugate (i.e., under $\mathrm{GL}(n, F)$*) to the span of the matrices $E_{i,n}$ with $i = 1, ..., n-1$, and in case b) $\mathcal{A}$ is $F-$conjugate to the span of the matrices $E_{n,i}$ with $i = 1, ..., n-1$.*

Proof. In either case, if $X, Y \in \mathcal{A}$ then $XY = 0$ and X has rank 1. For X of rank 1, we have $XF^n = Fy$ for some $y \neq 0$. Thus there exists $\mu \in (F^n)^*$

with $Xz = \mu(z)y = (y \otimes \mu)(z)$ for all z. We conclude that $\mathcal{A}$ has a basis over F of the form $X_i = y_i \otimes \mu_i$ for $i = 1, ..., n-1$.

We now assume that case a) is true (the argument for the other case is essentially the same). In case a), there exists $z \in \bar{F}^n$ such that

$$\{X_1(z), ..., X_{n-1}(z)\}$$

is linearly independent over $\bar{F}$. This implies that

$$\mu_1(z) \cdots \mu_{n-1}(z) y_1 \wedge \cdots \wedge y_{n-1} \neq 0.$$

Thus $y_1, ..., y_{n-1}$ are linearly independent. But $0 = X_i X_j = \mu_i(y_j) y_i \otimes \mu_j$. Thus $\mu_i(y_j) = 0$ for all $j = 1, ..., n-1$. Let ν be a non-zero element of $(F^n)^*$ such that $\nu(y_i) = 0$ for all $i = 1, ..., n-1$. Then ν is unique up to non-zero scalar multiple. Thus $y_i \otimes \nu$, $i = 1, ..., n-1$ is an F–basis of $\mathcal{A}$. There exists $g \in \mathrm{GL}(n, F)$ such that if $e_1, ..., e_n$ is the standard basis and $\xi_1, ..., \xi_n$ is the dual basis then $g y_i = e_i$ and $\nu \circ g = \xi_n$. This completes the proof in case a). □

Proposition 4.2. (Wallach) *Suppose that $\mathcal{L} \subset \mathcal{M}_n(F)$ is a subalgebra such that $\mathcal{L} \otimes_F \overline{F}$ is either:*

a) conjugate to the parabolic subalgebra $\mathcal{P}_{\bar{F}}$,

b) conjugate to the parabolic subalgebra $(\mathcal{P}_{\bar{F}})^{\mathrm{T}}$.

In case a) $\mathcal{L}$ is F–conjugate to $\mathcal{P}_F$. In case b) $\mathcal{L}$ is F–conjugate to $\mathcal{P}_F^{\mathrm{T}}$.

Proof. We just do case a) as case b) is proved in the same way. We look upon $\mathcal{L}$ as a Lie algebra over F. Then Levi's theorem [5, p. 91] implies that $\mathcal{L} = S \oplus R$ with S a semi-simple Lie algebra and R the radical (the maximal solvable ideal). Thus $\mathcal{L} \otimes_F \overline{F} = S \otimes_F \overline{F} \oplus R \otimes_F \overline{F}$. Therefore $R \otimes_F \overline{F}$ is the radical of $\mathcal{L} \otimes_F \overline{F}$. If we conjugate $\mathcal{L} \otimes_F \overline{F}$ to $\mathcal{P}_{\bar{F}}$ via $h \in \mathrm{GL}(n, \bar{F})$, then we see that

$$h[R \otimes_F \overline{F}, R \otimes_F \overline{F}]h^{-1}$$

has basis $E_{i,n}$, $i = 1, ..., n-1$. Thus hypothesis a) of Lemma 4.1 is satisfied for $\mathcal{A} = [R, R]$. There exists therefore $g \in \mathrm{GL}(n, F)$ such that $g\mathcal{A}g^{-1}$ has basis $E_{i,n}$, $i = 1, ..., n-1$. Assume that we have replaced $\mathcal{L}$ with $g\mathcal{L}g^{-1}$. Then $\mathcal{A}$ has basis $E_{i,n}$, $i = 1, ..., n-1$. Since $[\mathcal{L}, \mathcal{A}] \subset \mathcal{A}$ and $\mathcal{P}_F$ is exactly the set of elements X of $\mathcal{M}_n(F)$ such that $[X, \mathcal{A}] \subset \mathcal{A}$, we have $\mathcal{L} \subset \mathcal{P}_F$. Thus $\mathcal{L} = \mathcal{P}_F$, as both sides have the same dimension. □

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