

MIND SWITCHES IN FUTURAMA AND STARGATE

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Abstract

We generalize two mind-switching problems that arise in connection with the popular sci-fi television series *Futurama* and *Stargate SG-1*. Optimal solutions to these problems are found by answering the following question about a permutation σ expressed as a product of nontrivial disjoint cycles. “When writing σ as a product of distinct transpositions none equal to a factor of σ , what is the smallest number of transpositions that can be used?”

1 Introduction: Problems from *Futurama* and *Stargate*

“The Prisoner of Benda” [9], [7], a 2010 episode of *Futurama*, features a two-body mind-switching machine. Any pair can enter the machine to swap minds, but the machine has the limitation that it will not work more than once on the same pair of bodies. A two-body mind-switching machine with exactly the same limitation is featured in “Holiday” [4], a 1999 episode of *Stargate SG-1*. In both episodes, the participants in the mind swapping want to return to their original bodies, but they don’t know how. Brilliant characters (Sweet Clyde Dixon in *Futurama*, Captain Samantha Carter in *Stargate SG-1*) save the day by figuring out how to reverse the switching.

We first present a problem arising from *Futurama*. In “The Prisoner of Benda,” Amy and the Professor enter the machine and swap minds. Unhappy with their new bodies, they re-enter the machine and attempt to undo the switch. When nothing changes, the Professor recalls that due to cerebral immune response, “Once two bodies have switched minds, they can never switch back.” He proposes the idea of employing a third body for temporary mind storage, but he soon realizes that three bodies will not suffice to restore all minds to their original bodies. Agitated, Amy exclaims, “Oh no! Is it possible to get everyone back to normal using four or more bodies?” The Professor responds, “I’m not sure. I’m afraid we need to use...math!”

Throughout the episode, nine characters take part in a mind-swapping spree: Fry, Zoidberg, Amy, Hermes, Leela, Professor, Bender, Emperor, and Washbucket. For brevity, call their bodies 1, 2, 3, 4, 5, 6, 7, 8, and 9, respectively. These characters engage in a sequence of seven swaps: first the pair 3,6 swap minds, then 3,7, then 5,6, then 3,9, then 1,2, then 8,9, and finally 4,5.

After many adventures in each other’s bodies, the characters decide that they were better off before the switching. The Professor (in Bender’s body)

tries in vain to find a way to restore all minds to their original bodies. Sweet Clyde Dixon comes to the rescue with his proof of an inversion theorem displayed on a greenboard. The theorem shows how to undo the switching with a sequence of thirteen swaps, as given in (2.6) below. Sweet Clyde becomes a hero and the Emperor makes him a duke. Curiously though, the characters actually restore normalcy with a sequence of thirteen swaps *different* from that shown on the greenboard; first s and 1 swap minds, then the pair $t,2$, then $s,2$, then $t,1$, then $s,6$, then $t,9$, then $s,5$, then $t,8$, then $s,4$, then $t,7$, then $s,3$, then $t,6$, and finally $s,9$, where s and t denote the bodies of Globetrotters Sweet Clyde and Bubblegum Tate.

In a popular 2010 video [5], Cambridge University mathematician James Grime points out that the switching could be reversed without employing the bodies s and t , using the following nine swaps: first 2 and 3 swap minds, then the pair 1,9, then 1,8, then 1,7, then 1,6, then 1,5, then 1,4, then 1,3, and finally 2,9. Grime proceeds to ask if 9 is the *smallest* number of swaps that could accomplish the reversal. In the next section, this question is reformulated using mathematical terminology. Then in Theorem 3.4, we give an affirmative answer to a general version of the question involving an arbitrary number of bodies.

We next turn to a problem arising from *Stargate SG-1*. In “Holiday,” a crisis is created when the aging Machello tricks young handsome Daniel into swapping minds with him. As Jack and Teal’c attempt to help their colleague Daniel by retrieving the mind-switching machine, they blunder into swapping themselves. Unaware that Machello has designed the machine to block a direct reversal, physicist Samantha Carter attempts in vain to rectify the blunder by instructing Jack and Teal’c to grasp the machine’s handles in different ways. Meanwhile, Machello is enjoying his holiday in Daniel’s body.

Machello later feels guilty about what he has done to Daniel, and tells Samantha, “If I could trade places with him again, I would. But unfortunately, I am the only one who cannot.” His last sentence sparks Samantha’s idea on how to reverse the switching. She tells Jack, “We’re going to have to play a little musical chairs with your bodies.” At Samantha’s instruction, first the bodies of Teal’c and Daniel swap minds, then Jack and Machello, then Teal’c and Machello, and finally Jack and Daniel. This sequence of four swaps brings everyone back to normal, and Samantha earns the undying gratitude of the Stargate crew. Is 4 the *smallest* number of swaps that can accomplish the reversal? In Theorem 3.5, we give an affirmative answer to a general version of this question involving an arbitrary even number of

bodies.

2 Mind switching and permutations

In this section, we generalize the aforementioned problems and frame them in the language of group theory. Let S_n denote the group of $n!$ permutations of $\{1, 2, \dots, n\}$.

We begin with the problem from *Futurama*. Recall that the nine characters engage in a sequence of seven swaps, beginning with the pair 3,6. We represent this sequence by the right-to-left formal product

$$B_9 := (45) \cdot (89) \cdot (12) \cdot (39) \cdot (56) \cdot (37) \cdot (36), \quad (2.1)$$

where the factors in (2.1) are transpositions (2-cycles) in the symmetric group S_9 .

In the sequel, the symbol B (with or without a subscript) will always denote a formal product of distinct transposition factors in S_n , representing a sequence of swaps that one aims to reverse. For brevity, “transposition factors” will simply be called “factors”. When the factors of B are written in reverse order, the resulting formal product will be denoted by B^{-1} . The product B is an element in the group of words on the alphabet of transpositions.

One can view B as a permutation in S_n via right-to-left multiplication in S_n of its factors. This permutation will be denoted by $\sigma(B)$. Every permutation in S_n can be expressed as a product of nontrivial disjoint cycles [1, Theorem 2.3.5], and we will always assume that $\sigma(B)$ is expressed in this form. For example, the product $B = (12) \cdot (23)$ effects the 3-cycle $\sigma(B) = (123) \in S_3$. This 3-cycle represents the permutation that sends 1’s mind to 2, 2’s mind to 3, and 3’s mind to 1. From (2.1),

$$\sigma(B_9) = (12)(3456789). \quad (2.2)$$

The mind-switching history is encoded in the product B , but not generally in the permutation $\sigma(B)$. To illustrate this, consider the two products $B = (12) \cdot (23)$ and $B' = (23) \cdot (13)$. The product B tells us that the pair 1,2 swapped right after the pair 2,3, while B' tells us that 2,3 swapped right after 1,3. Although B and B' are completely different formal products, they effect the same permutation $\sigma(B) = \sigma(B') = (123)$. This permutation conveys

no information that distinguishes between the distinct switching histories encoded by B and B' .

The participants in *Futurama's* mind-swapping spree wish to undo B_9 ; that is, they want to reverse the switching in order to bring all minds back to their original bodies. To undo a general product of transpositions B representing a sequence of mind swaps, one needs to construct a formal product Q of distinct transpositions such that

- (i) the products Q and B have no factor in common; and
- (ii) $\sigma(Q) = \sigma(B)$.

Then $\sigma(Q^{-1} \cdot B)$ equals the identity permutation and Q^{-1} undoes B . To reverse the switching as efficiently as possible, we want to undo B using a product Q that has the smallest possible number of factors. Any such Q is said to be “optimal” or “best possible”. In the sequel, the symbol Q (with or without a subscript) will always denote a formal product of distinct transpositions.

For the product B_9 in (2.1), which represents the sequence of mind swaps in “The Prisoner of Benda,” Grime [5] observed that $\sigma(B_9) = \sigma(Q_9)$, where

$$Q_9 := (23) \cdot (19) \cdot (18) \cdot (17) \cdot (16) \cdot (15) \cdot (14) \cdot (13) \cdot (29). \quad (2.3)$$

The product Q_9 has no factor in common with the product B_9 . Thus Q_9^{-1} serves to undo B_9 using nine transpositions. Grime asked if Q_9 is best possible. We will give an affirmative answer to a more general question.

For $n \geq 5$, let B_n be any product of distinct transpositions representing a sequence of mind swaps with the following two properties:

- (i) bodies 1 and 2 swap with each other but not with any other body, as in (2.1); and
- (ii) as in (2.2),

$$\sigma(B_n) = (12)(345 \dots n). \quad (2.4)$$

Then analogous to the case $n = 9$ discussed in the preceding paragraph, we have $\sigma(B_n) = \sigma(Q_n)$, where

$$Q_n := (23) \cdot ((1n) \dots (15) \cdot (14) \cdot (13)) \cdot (2n). \quad (2.5)$$

The product Q_n has no factor in common with the product B_n . Thus Q_n^{-1} serves to undo B_n using n transpositions. Is Q_n best possible? For $n = 9$, this is Grime’s question. Theorem 3.4 gives an affirmative answer for each $n \geq 5$.

“The Prisoner of Benda” was written by Ken Keeler [10, p. 207], who earned a Ph.D. in applied mathematics at Harvard University in 1990. For the show, Keeler presented an algorithm [7] that is designed to reverse any sequence of swaps, i.e., it is designed to undo any product B . We gave a best possible algorithm optimizing Keeler’s in [3, Theorem 1]. Both algorithms depend only on the permutation $\sigma(B)$, not on the product B that effected $\sigma(B)$; in other words, both are independent of the switching history that effected the ultimate permutation. These algorithms require the entries in $\sigma(B)$ to switch only with bodies in the set $\{x, y\}$, where x and y are “outsiders” who did not participate in the initial mind-swapping spree. Because these algorithms are designed for situations where the switching history is forgotten, it should come as no surprise that they may not be optimal in situations where the switching history is known. For example, Keeler’s algorithm reverses B_9 with the product of thirteen transpositions given by

$$(x1) \cdot (y2) \cdot (x2) \cdot (y1) \cdot (x3) \cdot (x4) \cdot (x5) \cdot (x6) \cdot (x7) \cdot (x8) \cdot (y9) \cdot (x9) \cdot (y3), \quad (2.6)$$

while by (2.3), B_9 can be undone with only nine transpositions. We consider further comparisons of this kind in Section 5.

We next look at the problem from *Stargate SG-1*. For brevity, denote the bodies of Teal’c, Jack, Machello, and Daniel by 1, 2, 3, and 4, respectively. Recall that Daniel initially swapped minds with Machello, after which Teal’c swapped minds with Jack. This sequence of two mind swaps can be represented by the product

$$H_4 := (12) \cdot (34). \quad (2.7)$$

Samantha Carter observed that $\sigma(H_4) = \sigma(Q_4)$, where

$$Q_4 := (24) \cdot (13) \cdot (23) \cdot (14). \quad (2.8)$$

The products H_4 and Q_4 have no factor in common. Thus Q_4^{-1} serves to undo H_4 using four transpositions. It follows easily from parity considerations (as in Lemma 3.3 below) that H_4 cannot be undone with fewer than four transpositions, so that Q_4 is best possible.

The product in (2.7) is a special case of the product

$$H_{2r} := (12) \cdot (34) \cdots (2r - 1, 2r), \quad r \geq 1. \quad (2.9)$$

Observe that

$$\sigma(H_{2r}) := (12)(34) \cdots (2r-1, 2r), \quad r \geq 1, \quad (2.10)$$

since the factors of H_{2r} are already disjoint. We proceed to extend the problem of undoing H_4 to that of undoing H_{2r} . The switching history now is a sequence of r disjoint mind swaps represented by (2.9). We are interested in finding an optimal product Q for which Q^{-1} undoes H_{2r} . It turns out that such a product Q needs no outsiders, except in the case $r = 1$.

First consider the case $r = 1$. Using the outsiders 3 and 4, we have

$$\sigma(Q_2) = \sigma(H_2) = (12), \quad (2.11)$$

where

$$Q_2 := (34) \cdot (23) \cdot (14) \cdot (13) \cdot (24). \quad (2.12)$$

Since (12) is not a factor of the product Q_2 , it follows that Q_2^{-1} serves to undo H_2 using five transpositions. We can show directly that H_2 cannot be undone with fewer than five transpositions. For suppose that (12) were equal to a product of $u < 5$ distinct transpositions, none equal to (12). Then since (12) is an odd permutation, we would have $u = 3$ (again by Lemma 3.3), so that some product of four distinct transpositions would equal the identity. This is easily seen to be impossible.

Having already discussed the case $r = 2$ in the paragraph containing (2.8), we next consider the case $r = 3$. Observe that

$$\sigma(Q_6) = \sigma(H_6) = (12)(34)(56), \quad (2.13)$$

where

$$Q_6 := (15) \cdot (25) \cdot (35) \cdot (46) \cdot (45) \cdot (16) \cdot (13). \quad (2.14)$$

The product Q_6 has no factor in common with $H_6 = (12) \cdot (34) \cdot (56)$. Thus Q_6^{-1} serves to undo H_6 using seven transpositions.

For general $r > 3$, we can undo H_{2r} as follows. If r is even, undo the first two factors of H_{2r} using four transpositions as in (2.8), then repeat the process for the next two factors of H_{2r} , etc. In this way, we undo H_{2r} with $2r$ transpositions. If r is odd, undo the first three factors of H_{2r} using seven transpositions as in (2.14), then undo consecutive pairs of remaining factors as in the case where r is even. In this way, we undo H_{2r} with $2r + 1$

transpositions. In summary, for any $r \geq 2$, the product H_{2r} can be undone with $2r + \epsilon_r$ transpositions, where

$$\epsilon_r = \begin{cases} 0, & \text{if } 2 \mid r \\ 1, & \text{if } 2 \nmid r. \end{cases} \quad (2.15)$$

Is this best possible? Theorem 3.5 gives an affirmative answer for each $r \geq 2$. In the exceptional case $r = 1$, we've already seen that H_2 can be undone with five transpositions, and this is best possible.

The proofs of Theorems 3.4 and 3.5 depend on Theorem 3.1, which may be of independent interest because it answers the following question about a permutation σ expressed as a product of nontrivial disjoint cycles. "When writing σ as a product of distinct transpositions none equal to a factor of σ , what is the smallest number of transpositions that can be used?"

3 Optimality theorems

Let σ be a permutation expressed as a product of $m \geq 1$ nontrivial disjoint cycles, and let $n \geq 2$ denote the number of entries in σ . A paper by Mackiw [8] opens with this question: When writing σ as a product of transpositions, what is the smallest number of transpositions that can be used? The well-known answer is $n - m$; see [8], [6, Theorem 4].

The following theorem answers a refinement of this question when $n > 2$. For brevity, if a transposition τ appears as a factor in the disjoint cycle factorization of σ , we will simply say " τ occurs in σ ".

Theorem 3.1. *Let σ be a permutation expressed as a product of $m \geq 1$ nontrivial disjoint cycles, and let $n > 2$ denote the number of entries in σ . Suppose that $\sigma = \sigma(Q)$, where Q is a product of distinct transpositions, none occurring in σ . Then the smallest number M of transposition factors that Q can have is*

$$M = n - m + R + \epsilon_R, \quad (3.1)$$

where R is the number of transpositions occurring in σ , and ϵ_R is as defined in (2.15).

It is interesting to note that M depends only on the number of transposition factors, the number of cycles, and the number of entries in σ , but not on any additional information about the cycle structure of σ . For example,

either $\sigma = (12)(345)(678)(9abcde)$ or $\sigma = (1234)(56)(789a)(bcde)$ would yield the same value of M .

Theorem 3.1 does not hold for $n = 2$, since then $M = 5$ by the argument following (2.12). It can be shown that Theorem 3.1 would be valid for all $n \geq 2$ if we dropped the requirement that the factors of Q be distinct. In particular, for the case $n = 2$, note that $(12) = (23)(13)(23)$.

We prove Theorem 3.1 in the next section, using the following two well-known results from graph theory and group theory. Proofs of these results may be found in [2, Theorem 11.2.1, p. 163] and [1, pp. 82, 149], respectively.

Lemma 3.2. *A connected graph on N vertices has at least $N - 1$ edges.*

Lemma 3.3. *If the product of k transpositions in the symmetric group S_n equals the identity, then k must be even.*

Theorems 3.4 and 3.5 below solve the optimality problems arising from *Futurama* and *Stargate SG-1* that were posed in Section 2. To prove Theorem 3.4, we will apply Theorem 3.1 with σ chosen to be $\sigma(B_n)$. In this case, the lone transposition (12) occurring in σ is also a factor of B_n . To prove Theorem 3.5, we will apply Theorem 3.1 with σ chosen to be $\sigma(H_{2r})$. In this case, all transpositions occurring in σ are factors of H_{2r} . We are thereby able to exploit the switching histories that effected $\sigma(B_n)$ and $\sigma(H_{2r})$, even though no switching history is involved in Theorem 3.1 per se.

Theorem 3.4. *Let $n \geq 5$. Then the product B_n defined above (2.5) cannot be undone with fewer than n transpositions.*

Proof. Recall from (2.4) that $\sigma(B_n) = (12)(345 \dots n)$. Suppose that $\sigma(B_n) = \sigma(Q)$ for some product Q of distinct transpositions, none equal to a factor of B_n . We need to show that Q has at least n factors. By definition, B_n has the factor (12); thus the product Q cannot have (12) as a factor. It follows that Q is a product of distinct transpositions none occurring in $\sigma(B_n)$. We can therefore apply Theorem 3.1 with $\sigma = \sigma(B_n)$ to conclude that the number of factors in Q is at least

$$M = n - m + R + \epsilon_R = n - 2 + 1 + 1 = n.$$

□

Theorem 3.5. *Let $r \geq 2$. Then the product H_{2r} defined in (2.9) cannot be undone with fewer than $2r + \epsilon_r$ transpositions.*

Proof. Suppose that $\sigma(H_{2r}) = \sigma(Q)$ for some product Q of distinct transpositions, none equal to a factor of H_{2r} . Therefore Q is a product of distinct transpositions none occurring in $\sigma(H_{2r})$. Applying Theorem 3.1 with $\sigma = \sigma(H_{2r})$ and $R = r$, we conclude that the number of factors in Q is at least

$$M = n - m + r + \epsilon_r = 2r - r + r + \epsilon_r = 2r + \epsilon_r.$$

□

4 Proof of Theorem 3.1

With $R = r$, we may express the permutation σ in Theorem 3.1 as the following product of disjoint cycles:

$$\sigma = \sigma(H_{2r})C_1 \cdots C_{m-r}, \quad (4.1)$$

where H_{2r} is defined in (2.9) and the C_i are disjoint cycles of length $\ell_i > 2$. (If $m = r$, then σ is interpreted as $\sigma(H_{2r})$.) Our object is to prove (3.1). We may suppose that $r \geq 1$, since if no transposition occurs in σ , then $M = n - m$ as in [8].

We proceed to show that σ has parity $n - m + r + \epsilon_r$, by which we mean that the permutation σ is even or odd according as the integer $n - m + r + \epsilon_r$ is even or odd. Each C_i equals a product of $\ell_i - 1$ transpositions; for example, $(abcdef) = (ab)(bc)(cd)(de)(ef)$. Therefore $C_1 \cdots C_{m-r}$ has parity

$$\sum_{i=1}^{m-r} (\ell_i - 1) = (n - 2r) - (m - r) = n - m - r.$$

Since $\sigma(H_{2r})$ is a product of r transpositions, it follows from (4.1) that σ has parity $n - m$. Equivalently, σ has parity $n - m + r + \epsilon_r$, since $r + \epsilon_r$ is even by (2.15).

Our object is to prove that $M = n - m + r + \epsilon_r$ for $n > 2$, $r \geq 1$. We begin by showing that

$$M \leq n - m + r + \epsilon_r. \quad (4.2)$$

To accomplish this, it must be shown that $\sigma = \sigma(Q)$ for some product Q of $n - m + r + \epsilon_r$ distinct transpositions, none occurring in σ .

First consider the case $r = 1$. By (2.4)–(2.5), the permutation $(12)C_1$ equals a product of $2 + \ell_1$ distinct transpositions, none equal to the transposition (12) . Thus $\sigma = (12)C_1 \cdots C_{m-1}$ equals a product of

$$2 + \ell_1 + \sum_{i=2}^{m-1} (\ell_i - 1) = n - m + 2$$

distinct transpositions, none equal to (12) . This proves (4.2) in the case $r = 1$.

Next let $r \geq 2$. The statement containing (2.15) tells us that for $r \geq 2$, $\sigma(H_{2r})$ equals a product of $2r + \epsilon_r$ distinct transpositions, none equal to a factor of H_{2r} . Thus $\sigma = \sigma(Q)$ for a product Q of

$$2r + \epsilon_r + \sum_{i=1}^{m-r} (\ell_i - 1) = n - m + r + \epsilon_r$$

distinct transpositions, none occurring in σ . This completes the proof of (4.2). (Note that Q has been constructed such that every entry in Q is an entry in σ . This is in contrast to the case $n = 2$, as we see from (2.12)).

It remains to prove the reverse inequality

$$M \geq n - m + r + \epsilon_r. \tag{4.3}$$

We need only prove

$$M \geq n - m + r; \tag{4.4}$$

indeed, (4.3) and (4.4) are equivalent by Lemma 3.3, since as was noted above, σ has parity $n - m + r + \epsilon_r$. Supposing that $\sigma = \sigma(Q)$ for a product Q of w distinct transpositions, none occurring in σ , our goal is now to show that $w \geq n - m + r$.

To illuminate the exposition, we will use the running example

$$\sigma = (12)(34)(56)(78)(9ab)(cde) = \sigma(Q),$$

with Q equal to the product

$$(15) \cdot (89) \cdot (25) \cdot (7b) \cdot (4x) \cdot (35) \cdot (7a) \cdot (46) \cdot (79) \cdot (45) \cdot (8b) \cdot (16) \cdot (13) \cdot (cd) \cdot (de) \cdot (3x),$$

where a, b, c, d, e, x stand for 10, 11, 12, 13, 14, 15. In this example, $n = 14$, $r = 4$, $m = 6$, and $w = 16$.

Let G be a graph whose vertex set $V(G)$ is the set of entries in Q , and whose edges $[ij]$ correspond to the w factors (ij) of Q . The graph G for the product Q in our example is illustrated in Figure 1. Since $\sigma = \sigma(Q)$, the set $V(G)$ contains all the entries in σ , but $V(G)$ may also contain entries in Q outside of σ . In our example, x is the only entry in Q outside of σ .

For each i with $1 \leq i \leq m-r$, G has a connected component G_i such that $V(G_i)$ contains all of the entries in the cycle C_i . Let J denote the union of the components G_i . (View J as empty if $m = r$.) In our example, J is the union of two components G_1, G_2 with $V(G_1) = \{7, 8, 9, a, b\}$, $V(G_2) = \{c, d, e\}$, so $V(J) = \{7, 8, 9, a, b, c, d, e\}$.

The set $\{1, 2, \dots, 2r\} \subset V(G)$ can be written as a disjoint union

$$\{1, 2, \dots, 2r\} = A \cup Z, \quad (4.5)$$

where $A \subset V(J)$ and Z is disjoint from $V(J)$. For the cardinalities, write $\alpha := |A|$, so that $|Z| = 2r - \alpha$. In our example, $A = \{7, 8\}$ with $\alpha = 2$ and $Z = \{1, 2, 3, 4, 5, 6\}$.

Since $V(J)$ contains the $n - 2r$ entries in the cycles C_i , we have $|V(J)| \geq n - 2r + \alpha$. By Lemma 3.2, each G_i contains at least $|G_i| - 1$ edges. Therefore, since the graph J is the disjoint union of at most $m - r$ distinct components G_i , the number of edges in J is at least

$$(n - 2r + \alpha) - (m - r) = n - m + \alpha - r.$$

It follows that Q has at least $n - m + \alpha - r$ distinct transposition factors whose entries are all in $V(J)$. In particular, $w \geq n - m + \alpha - r$, but this does not quite accomplish our goal of showing that $w \geq n - m + r$. Thus we seek additional transposition factors of Q , this time with entries *outside* of $V(J)$.

With the aim of obtaining these additional factors of Q , we will construct a set E with $Z \subset E \subset V(G)$ such that E is disjoint from $V(J)$ and the elements of E fill at least $2|E|$ slots in Q . (Note that any element of E not in Z is necessarily outside of σ .) Since $|E| \geq |Z|$, we will obtain the desired lower bound

$$w \geq (n - m + \alpha - r) + |E| \geq (n - m + \alpha - r) + |Z| = n - m + r.$$

If each of the elements in Z fills at least two slots in Q , then we may take $E = Z$ and the proof is complete. It remains to construct E in the case where Z contains “singletons,” i.e., elements that occur only once as entries

in Q . Note that a singleton corresponds to a vertex of degree 1 in $V(G)$. In our example, 2 is the only singleton in $Z = \{1, 2, 3, 4, 5, 6\}$.

We proceed to specify the elements that will make up the desired set E defined in (4.6) below. A singleton $b_1 \in Z$ is paired in Q with some $g \in V(G)$, that is, Q has a factor (gb_1) . In our example, $(gb_1) = (25)$. Note that g cannot be a singleton, otherwise (gb_1) would be a factor of Q that occurs in σ . Since $b_1 \notin V(J)$, we have $g \notin V(J)$. Let $S(g) := \{b_1, \dots, b_k\}$ be the set of all singletons in Z that are paired with g , and let

$$(gb_1), (gb_2), \dots, (gb_k)$$

be transpositions appearing in that same left-to-right order among the factors of Q . In our example, $k = 1$.

Since no factor of Q occurs in σ and $b_1 \in Z$, the permutation σ cannot map the singleton b_1 to g . Thus g occurs as an entry in Q to the left of the factor (gb_1) . Similarly, σ cannot map g to the singleton b_k , so g occurs as an entry in Q to the right of (gb_k) . Therefore g occurs at least $k + 2$ times as an entry, so that all together, the $k + 1$ elements in

$$U(g) := \{b_1, \dots, b_k, g\} = S(g) \cup \{g\}$$

fill at least $2(k + 1) = 2|U(g)|$ slots in Q . In our example, (15) occurs to the left of the factor $(gb_1) = (25)$ in Q , both (35) and (45) occur to the right of (25), and the two elements in $U(5) = \{2, 5\}$ together fill five slots in Q .

If Z contains another singleton $b'_1 \notin S(g)$, then repeat this procedure with an element $g' \in V(G)$ that is paired with b'_1 . Clearly $g' \neq g$. Note that $S(g')$ is disjoint from $S(g)$, since an element common to both sets would have to occur at least twice as an entry in Q . Thus $U(g')$ is disjoint from $U(g)$. Repeat the procedure again and again until all the singletons in Z have been exhausted.

Let Z^* denote the set of all non-singletons in Z which are not in the set $\{g, g', \dots\}$. The elements of Z^* fill at least $2|Z^*|$ slots in Q . Define E to be the disjoint union

$$E = Z^* \cup U(g) \cup U(g') \cup \dots \quad (4.6)$$

In our example, $E = Z$ (but E would strictly contain Z in an example where the set $\{g, g', \dots\}$ contained an element outside of σ). The set E in (4.6) satisfies the required conditions, since

$$E = Z \cup \{g, g', \dots\} \subset V(G),$$

E is disjoint from $V(J)$, and the elements of E fill at least $2|E|$ slots in Q . \square

5 Conclusion and related problems

Let $\sigma(B)$ be a known permutation in S_n corresponding to a product B representing a (possibly forgotten) switching history. This section begins with a summary of optimal ways for undoing B when

- (i) B is unknown,
- (ii) $B = H_{2r}$ (defined in (2.9)),
- (iii) $B = B_n$ (defined above (2.5)).

We then discuss the situation for some other classes of products B . Our examples illustrate how dramatically different the optimal solutions can be for products B and B' representing different switching sequences, even when B and B' effect the same permutation $\sigma(B) = \sigma(B')$.

Assume that $\sigma(B)$ has n entries and m nontrivial disjoint cycles. Let $j(B)$ denote the number of factors in an optimal product Q for which Q^{-1} undoes B . If B represents a *forgotten* switching history, then by [3, Theorem 1],

$$j(B) = n + m + 2, \quad (5.1)$$

and Q requires two outsiders. In cases where there is full information about B , we can often reduce the right side of (5.1). For example, if $B = H_n = H_{2r}$ with $r \geq 2$, then Theorem 3.5 and the statement containing (2.15) show that

$$j(B) = n \text{ or } j(B) = n + 1 \text{ according as } r \text{ is even or odd,}$$

and no outsiders are needed. Even if only partial information is available about B , for instance when $B = B_n$, we may be able to reduce the right side of (5.1). For example, Theorem 3.4 and (2.5) show that

$$j(B_n) = n, \quad \text{for } n \geq 5, \quad (5.2)$$

and no outsiders are needed.

Let B'_n denote the product Q_n defined in (2.5). Then $\sigma(B'_n) = \sigma(B_n)$. We can reduce the right side of (5.2) upon replacing B_n by B'_n , as follows:

$$j(B'_n) = n - 2, \quad \text{for } n \geq 5. \quad (5.3)$$

The idea behind (5.3) is to undo B'_n with the product

$$(12) \cdot (3n) \cdot (4n) \cdots (n-1, n).$$

One can show that this way of undoing B'_n is best possible, by using Lemma 3.3 and [3, Lemma 1(i),(ii)]. For an example with $n = 5$, we have

$$B'_5 = (23) \cdot (15) \cdot (14) \cdot (13) \cdot (25) \quad \text{and} \quad \sigma(B'_5) = \sigma(B_5) = (12)(345),$$

and $j(B'_5) = 3$, since $(12)(35)(45)$ undoes B'_5 .

Yet another product B with $\sigma(B) = (12)(345)$ is

$$B = (25) \cdot (45) \cdot (35) \cdot (14) \cdot (24).$$

Since $\sigma(B) = \sigma(Q)$ for $Q := (23) \cdot (12) \cdot (15) \cdot (13) \cdot (34)$, Q^{-1} serves to undo B with five transpositions. We leave it as an exercise for the reader to show that this is best possible, i.e., $j(B) = 5$.

We proceed to give the values of $j(B)$ for three different classes of products B which effect the same n -cycle $\sigma(B) = (123\dots n)$. Again these examples illustrate how sensitive the optimal solutions can be to the switching history. It would be interesting to find $j(B)$ for other natural classes of products B , and to determine how many outsiders are required for undoing these B .

(a) If

$$B = (12) \cdot (23) \cdot (34) \cdots (n-1, n) \quad \text{with} \quad n \geq 5,$$

then $\sigma(B) = (123\dots n)$ and B can be undone with $n + 1$ transpositions, without outsiders. Moreover, by [3, Theorem 2], $n + 1$ is best possible, i.e., $j(B) = n + 1$.

(b) If

$$B = (n, n-1) \cdots (n3) \cdot (n2) \cdot (n1) \quad \text{with} \quad n \geq 3,$$

then $\sigma(B) = (123\dots n)$ and B can be undone with $n + 1$ transpositions, where one outsider is necessary and sufficient. Moreover, by [3, Theorem 3], $n + 1$ is best possible, i.e., $j(B) = n + 1$.

(c) In (a) and (b) above, B has only $n - 1$ factors. If B is a product of all $(n^2 - n)/2$ transpositions in S_n , then two outsiders are necessary and sufficient to undo B , and by [3, Theorem 1], $j(B) = n + m + 2$. As shown in [3], when say n is congruent to 1 modulo 4, it is not hard to give a recipe for constructing such B in S_n with $\sigma(B) = (123\dots n)$. For an example in S_5 , consider the product of all ten transpositions

$$B = (54) \cdot (53) \cdot (52) \cdot (51) \cdot (12) \cdot (23) \cdot (14) \cdot (13) \cdot (24) \cdot (34);$$

then $\sigma(B) = (12345)$ and $j(B) = 8$.

Examples of two products B and B' which effect the same 4-cycle $\sigma(B) = \sigma(B') = (1234)$ are

$$B = (24) \cdot (23) \cdot (14) \quad \text{and} \quad B' = (23) \cdot (12) \cdot (34) \cdot (14) \cdot (24).$$

It is easy to see that $(34) \cdot (12) \cdot (13)$ undoes B and $j(B) = 3$. We close with another fun problem for the reader: show that two outsiders are necessary and sufficient to undo B' , and $j(B') = 7$.

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