NONEXISTENCE OF ODD PERFECT NUMBERS OF A CERTAIN FORM

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ABSTRACT

Write $N = p^{\alpha} q_1^{2\beta_1} \cdots q_k^{2\beta_k}$, where $p, q_1, \ldots, q_k$ are distinct odd primes and $p \equiv \alpha \equiv 1 (\bmod 4)$. An odd perfect number, if it exists, must have this form. McDaniel proved in 1970 that N is not perfect if all β_i are congruent to $1 (\bmod 3)$. Hagis and McDaniel proved in 1975 that N is not perfect if all β_i are congruent to $17 (\bmod 35)$. We prove that N is not perfect if all β_i are congruent to $32 (\bmod 65)$. We also show that N is not perfect if all β_i are congruent to $2 (\bmod 5)$ and either $7|N$ or $3|N$. This is related to a result of Iannucci and Sorli, who proved in 2003 that N is not perfect if each β_i is congruent either to $2 (\bmod 5)$ or $1 (\bmod 3)$ and $3|N$.

1. INTRODUCTION

Write

$$N = p^{\alpha} q_1^{2\beta_1} \cdots q_k^{2\beta_k}, \tag{1.1}$$

where $p, q_1, \ldots, q_k$ are distinct odd primes, $\alpha, \beta_1, \ldots, \beta_k \in \mathbb{N}$, and $p \equiv \alpha \equiv 1 (\bmod 4)$. Euler proved that an odd perfect number, if it exists, must have the form (1.1). Let $\mathcal{O}$ denote the set of odd perfect numbers. In the case $\beta_1 = \cdots = \beta_k = \beta$, Hagis and McDaniel [3, p. 27] conjectured that $N \notin \mathcal{O}$. This conjecture was already proved for $\beta = 1$ in 1937 [7] and for $\beta = 2$ in 1941 [5]. More recently, the conjecture has been proved for some larger values of β, including $\beta = 3, 5, 6, 8, 11, 12, 14, 17, 18, 24$, and 62 (see [1]). We now describe some infinite classes of β for which the conjecture is known to hold. Write

$$\gamma_i := 2\beta_i + 1, \quad 1 \leq i \leq k. \tag{1.2}$$

The assertion

$$d | \gamma_i \quad \text{for all } i \Rightarrow N \notin \mathcal{O} \tag{1.3}$$

was proved for $d = 3$ by McDaniel [6] in 1970, and for $d = 35$ by Hagis and McDaniel [3] in 1975. In particular, this proves the conjecture for the infinite classes $\beta \equiv 1 (\bmod 3)$ and $\beta \equiv 17 (\bmod 35)$.

In Theorem 2 (see Section 3), we prove (1.3) for $d = 65$, which in particular proves the conjecture for all $\beta \equiv 32 (\bmod 65)$. When d is a product of two primes > 3, the only values of d for which (1.3) is known are now $d = 35, 65$. There are no prime values $d > 3$ for which (1.3) is known.

Recently, Iannucci and Sorli [4] extended the result of McDaniel [6] by proving that

$$(3 | N \text{ and } \gcd(\gamma_i, 15) > 1 \text{ for all } i) \Rightarrow N \notin \mathcal{O}. \tag{1.4}$$

(This has an important application to bounds for the number of prime factors in odd perfect numbers.) We can prove the following related results:

$$(3 | N \text{ and } 7 | \gamma_i \text{ for all } i) \Rightarrow N \notin \mathcal{O}, \tag{1.5}$$

$$(7 | N \text{ and } 5 | \gamma_i \text{ for all } i) \Rightarrow N \notin \mathcal{O}, \tag{1.6}$$

$$(5 | N \text{ and } 77 | \gamma_i \text{ for all } i) \Rightarrow N \notin \mathcal{O}, \tag{1.7}$$

$$(3 | N \text{ and } 143 | \gamma_i \text{ for all } i) \Rightarrow N \notin \mathcal{O}, \tag{1.8}$$

$$(13|N \text{ and } 55|\gamma_i \text{ for all } i) \Rightarrow N \notin \mathcal{O}. \tag{1.9}$$

Of the last five assertions, we prove here only (1.6); see Theorem 1. Our proofs, like the proofs of McDaniel et al., depend on the following result of Kanold [5]:

$$(N \in \mathcal{O} \text{ and } d|\gamma_i \text{ for all } i) \Rightarrow d^4|N. \tag{1.10}$$

2. PRELIMINARIES

Let $\sigma(n)$ denote the sum of the positive divisors of n. Assume for the purpose of contradiction that $N \in \mathcal{O}$, so that, as in [4, eq.(2)],

$$2N = \sigma(N) = \sigma(p^\alpha) \prod_{i=1}^{k} \sigma(q_i^{2\beta_i}). \tag{2.1}$$

Define, for prime q and integer $d > 1$,

$$f(q) := f_d(q) = \sigma(q^{d-1}) = (q^d - 1)/(q - 1) \tag{2.2}$$

and

$$h(q) := h_d(q) = \sigma(q^{d-1})/q^{d-1}. \tag{2.3}$$

If $d|\gamma_i$ for all i, then for all i,

$$f_d(q_i) \;\text{ divides }\; f_{\gamma_i}(q_i), \tag{2.4}$$

so $f_d(q_i)$ divides N by (2.1) - (2.2). Since α is odd,

$$(p+1)/2 \;\text{ divides }\; \sigma(p^\alpha), \tag{2.5}$$

so $(p+1)/2$ divides N by (2.1). As in [4, p. 2078], it is easily seen that for odd primes $r > q$ and integers a, b, c with $a > 1,\; c > b > 1$,

$$h_c(q) > h_b(q) > h_a(r) \geq (r+1)/r. \tag{2.6}$$

Moreover, for odd prime $u \leq p$,

$$h_a(u)(p+1)/p \geq h_a(p)(u+1)/u, \tag{2.7}$$

since $h_a(x)^{-1}(x+1)/x$ is an increasing function in x for $x > 1$.

Let S denote the set of prime divisors of N. Suppose that $d|\gamma_i$ for all i. Then by (2.1) and (2.6),

$$2 = \frac{\sigma(N)}{N} = \frac{\sigma(p^\alpha)}{p^\alpha} \prod_{i=1}^{k} h_{\gamma_i}(q_i) \geq \frac{p+1}{p} \prod_{i=1}^{k} h_d(q_i) = \frac{p+1}{p} \prod_{\substack{s\in S \\ s\neq p}} h_d(s). \quad (2.8)$$

Let T be any subset of S containing a prime u satisfying the condition that $u \leq p$ if $p \in T$. We claim that

$$\frac{p+1}{p} \prod_{\substack{s\in S \\ s\neq p}} h_d(s) \geq \frac{u+1}{u} \prod_{\substack{t\in T \\ t\neq u}} h_d(t). \quad (2.9)$$

In the case $p \notin T$, (2.9) follows because

$$\prod_{\substack{s\in S \\ s\neq p}} h_d(s) \geq \prod_{t\in T} h_d(t) \geq \frac{u+1}{u} \prod_{\substack{t\in T \\ t\neq u}} h_d(t);$$

in the case $p \in T$, (2.9) follows from (2.7).

Our objective is to find a set $T = T(d, u)$ as above such that

$$\frac{u+1}{u} \prod_{\substack{t\in T \\ t\neq u}} h_d(t) > 2. \quad (2.10)$$

In view of (2.8) - (2.9), this will provide the desired contradiction to the assumption that $N \in \mathcal{O}$.

3. THEOREMS AND PROOFS

We begin with a lemma. Recall that S is the set of prime divisors of N.

Lemma. *If* $N \in \mathcal{O}$ *and* $13|\gamma_i$ *for all* i *and* $\gcd(p+1, 21) = 1$*, then* $13 \in S$ *and* $W \subset S$*, where*

$$W = \{53, 79, 131, 157, 313, 443, 521, 547, 677, 859, 911, 937, 1093, 1171, 1223, 1249, 1301, 1327, 1483, 1613, 1847\}$$

is the set of primes $\equiv 1 (\operatorname{mod} 13)$ *less than* 1850.

Proof. By (1.10) with $d = 13$, we have $\mathbf{13} \in S$. (Bold font is used to keep track of primes confirmed to lie in S.)

A list of primes

$$r_1, r_2, \ldots, r_n \tag{3.1}$$

is called a *d-chain* (or simply a *chain*) if $r_1 \in S$ and $r_{i+1}|f_d(r_i)$ for each $i < n$, where f_d is defined in (2.2). In this proof, we take $f = f_d$ with $d = 13$. If $r_i \neq p$ for each $i < n$, then every prime in the chain (3.1) lies in S, by (2.4). An example of a chain is

$$13, 264031, (882..981), \mathbf{79}. \tag{3.2}$$

Here (882..981) is a 64-digit prime whose center digits can be easily retrieved by factoring $f(264031)$. By hypothesis, the first and third primes in (3.2) cannot be p, because they are $\equiv 6 (\operatorname{mod} 7)$. The second and fourth primes cannot be p since they are $\equiv 3 (\operatorname{mod} 4)$. We know $13 \in S$, so $264031 \in S$ because $264031|f(13)$. Similarly, $(882..981) \in S$ since $(882..981)|f(264031)$. Finally, $79|f((882..981))$, so the chain (3.2) confirms that $79 \in S$.

None of the following chains can have p preceding its terminal prime r_n, and so each chain confirms that r_n (in bold) lies in S:

13, **53**;

13, 264031, (882..981), **157**;

79, (551..681), **1249**;

79, (551..681), 50909, **499903**;

499903, **1483**;

499903, 32579, (**313** and **937**);

937, **599**;

599, 847683(**443** and **1613**);

599, 45137, 6397, (**677** and **911**);

937, (111..851), 14561, **42304159**;

42304159, **3251**;

42304159, (766..419), (46073), (976..861), **859**;

3251, **131**;

1483, (301..587), **1223**;

1223, 920011, **2081**;

2081, (**547** and **1171**);

157, (281..937), 5669, 168247, (395..237), **1327**;

859, (183..471), **2029**;

499903, 32579, (**468**..**021**);

Next consider the pair of chains

$$\begin{cases} 313, (240..891), 9907, 1847; \\ 1249, (555..427), \mathbf{1847}; \end{cases}$$

The two chains in the pair have no common primes except the terminal prime 1847. Thus, while p might precede 1847 somewhere in one chain or the other, p cannot precede 1847 in both chains. Hence (at least) one chain in the pair does not have an occurrence of p preceding 1847, and that chain confirms that

$1847 \in S$. We now can form the single chains

$$1847, \mathbf{521};$$
$$521, (317..359), \mathbf{1951};$$
$$1951, (193..027), 4759, \mathbf{1301};$$

It remains to show that $1093 \in S$. This is accomplished with the following pair of chains:

$$\begin{cases} 2029, 65677, 18038593, 1093; \\ (468..021), 138581, (648..279), (112..139), 1873, (110..713), (582..641), \\ \quad (578..461), \mathbf{1093}; \end{cases}$$

□

Theorem 1. *Suppose that* $5|\gamma_i$ *for all* i*, and* $N \in \mathcal{O}$*. Then* $\gcd(N, 21) = 1$ *and* $p \equiv 1 (\text{mod}\, 12)$.

Proof. By (1.10) with $d = 5$, we have $\mathbf{5} \in S$.

Suppose for the purpose of contradiction that $p \equiv 2 (\text{mod}\, 3)$. Then by (2.5), $3 \in S$. As in (2.2), write $f = f_d$ with $d = 5$. Since $f(3) = 11^2$, (2.4) implies that $11 \in S$. Since $5|\gamma_i$ for all i and $5^4|N$ by (1.10), then, in the notation of (2.3) with $d = 5$, we obtain the contradiction

$$2 = \sigma(N)/N > h(3)h(5)h(11) > 2.05 \; . \tag{3.3}$$

This proves that $p \equiv 1 (\text{mod}\, 12)$.

We have seen that $5 \in S$. We now confirm additional primes in S by using d-chains as in the Lemma, but with $d = 5$ instead of $d = 13$. The chains

$$5, (\mathbf{11} \text{ and } \mathbf{71});$$
$$11, 3221, (195..931), \mathbf{41};$$

confirm that $11, 71$, and 41 lie in S, since neither 5 nor 3221 can equal p (as $p \equiv 1 (\text{mod}\, 12)$). Employing many such chains, we can construct a large set Y of primes in S consisting of 5 together with most of the primes $\equiv 1 (\text{mod}\, 5)$ which are $< 10^4$. The set Y and the long list of chains used to construct Y may be found at [2].

Suppose that $7|N$. With $T = Y \cup \{7\}$, we arrive at the contradiction (2.10) with $u = 61, d = 5$. Thus $7 \nmid N$. The same argument shows that $3 \nmid N$ (alternatively, $3 \nmid N$ follows from (1.4)). This completes the proof of Theorem 1. □

Theorem 2. *If* $65|\gamma_i$ *for all* i*, then* $N \notin \mathcal{O}$*.*

Proof. Assume for the purpose of contradiction that $65|\gamma_i$ for all i and $N \in \mathcal{O}$. From (1.10), we know that $13 \in S$. Let Y be as in the proof of Theorem 1, and let W be as defined in the Lemma. In view of Theorem 1, the hypotheses of the Lemma are satisfied, and so $Y \cup W \subset S$. With

$$T = Y \cup W \cup \{13\},$$

we obtain the desired contradiction (2.10) with $u = 61, d = 65$. This completes the proof of Theorem 2. □

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