Math 120A　　Final　　152 points　　March 22, 2013

Directions: Justify all answers. If you appeal to a theorem, show that the hypotheses of that theorem are justified. An integral $\int_{|z|=r}$ is interpreted to go once *counterclockwise* around the given circle.
Problems 1, 4 are worth 26 points each; problems 2, 3, 5, 6, 7 are worth 20 points each.

(1) (A) Find the residues of $f(z) = \dfrac{1}{z - z^3}$ at 0, at 1, and at ∞.
(B) For f as above, evaluate

$$\int_{|z|=2} f(z)dz.$$

SOLUTION: The Laurent series for $f(z)$ about 0, 1, and -1 are:
$1/z + z + z^3 + \ldots,$
$(-1/2)/(z-1) + 3/4 - (7/8)(z-1) + \ldots,$ and
$(-1/2)/(z+1) - 3/4 - (7/8)(z+1) + \ldots$.

Thus the residues at 0, 1, and -1 are 1, $-1/2$, and $-1/2$, respectively. Moreover, $f(1/z)/z^2 = z/(z^2-1)$ is analytic, so its residue at 0 equals 0. Thus the residue of $f(z)$ at ∞ equals 0. This last fact alone shows that the answer to part (B) is 0.

(2) Anna said: "If $g(z)$ has an antiderivative in the annulus $1 < |z| < 3$, then

$$\int_{|z|=2} g(z)dz = 0,$$

and consequently, since $1/z$ has an antiderivative $\log z$, we can conclude that

$$\int_{|z|=2} \frac{dz}{z} = 0."$$

Show that Anna's conclusion is false, and also discuss where her logic first broke down.

SOLUTION: Anna's conclusion is false because the integral equals $2\pi i$ by the residue theorem. The logic first broke down when she said that $1/z$ has an antiderivative $\log z$. Note that $\log z$ is not even continuous on the circle, let alone differentiable.

(3) Let $|z| < 1$. Write down the Taylor series (about 0) for $1/(1+z)$ and then integrate to derive the Taylor series (about 0) for $\text{Log}\,(1+z)$. Carefully justify every step.

SOLUTION: $1/(1+z) = \sum_{k=0}^{\infty} (-1)^k z^k$. Integrate along a straight line joining 0 to some point z with $|z| < 1$, to get
$\text{Log}(1+z) = \sum_{k=0}^{\infty} (-1)^k z^{k+1}/(k+1)$. We are allowed to integrate term by term as long as $|z| < 1$, and the path was chosen so that every integral could be evaluated using the antiderivative of the integrand.

(4) For each of the four values $k = 0, 1, 2, 3$, evaluate the integral

$$\int_{|z|=1} \frac{dz}{z^{k-1} \sin z},$$

and justify.

SOLUTION: $z/\sin z = 1/(1 - z^2/6 + \dots)$ is analytic for $|z| < \pi$. Using derivative formulas for the coefficients, or alternatively simply dividing the denominator into the numerator 1, we get the Maclaurin expansion $z/\sin z$ $= (1 + z^2/6 + \dots)$. Thus when we divide by z^k for $k = 0, 1, 2, 3$, we get the residues (at 0) equal to $0, 1, 0, 1/6$, respectively. The corresponding integrals thus equal (by the residue theorem) $0, 2\pi i, 0, \pi i/3$, respectively.

(5) Let $h(z) = 1 - z^3$ for $|z| \le 1$.
(A) Prove that for $|z| \le 1$, we have $|h(z)| \le 2$.
(B) Find all z with $|z| \le 1$ for which $|h(z)| = 2$.

SOLUTION: Part (A) follows from the triangle inequality, since $|z^3| = |z|^3 \le 1$. For part (B), note that that h attains its maximum value 2 when

$z = -1$. But is this the only answer for z? By the maximum modulus principle, the maximum value 2 of $|h|$ can only occur at certain points z of the form $z = \exp(i\theta)$. We want to find θ such that $4 = |h(z)|^2 = 1+|z|^6 - 2\Re z^3 = 2 - 2\cos(3\theta)$. Thus we want to solve $\cos(3\theta) = -1$. The solution is $\theta = d\pi/3$ where d is any odd integer. Thus $|h| = 2$ when z is either -1 or $\exp(\pm i\pi/3)$.

(6) Suppose that f is entire. For fixed u with $|u| < 2$, prove that as $N \to \infty$,

$$u^N \int_{|z|=2} \frac{f(z)dz}{(z-u)z^N} \quad \to \quad 0.$$

SOLUTION: See top of page 192.

(7) For nonzero z with $|\operatorname{Arg} z| < \pi$, explain in detail how you know that $f(z) = \operatorname{Log} z$ is analytic, and find the derivative f'.

SOLUTION: See page 95.