Math 120A Test 2 100 points March 1, 2013

Directions: Justify all answers. If you appeal to a theorem, show that the hypotheses of that theorem are justified. Avoid messy computations; look for elegant solutions. Each problem is worth 20 points.

(1) Let L be the horizontal line segment of length 23 which starts at $3i$ and ends at $3i + 23$. Show that $|I| < 1$, where

$$I := \int_L \frac{dz}{z^3 + 3i}.$$

SOLUTION: The denominator of the integrand has absolute value greater or equal to $|z|^3 - 3$, which is greater or equal to $3^3 - 3 = 24$ when z is on L. Thus the integrand has absolute value less than or equal to $1/24$, so $|I| \leq 23/24 < 1$.

(2) Anna claims that the function $\sin(\cos(z))$ is bounded on $\mathbb{C}$. Explain carefully how you know that Anna is wrong.

SOLUTION: The composite of two analytic functions is analytic. Thus if Anna were correct, then this composite function would be constant, by Liouville's Theorem. However, the function has different values at $z = 0$ and $z = \pi/2$, so it is not constant. Thus Anna is wrong.

(3) Let T be a triangle in $\mathbb{C}$ containing the point i in its interior. Explain in detail why

$$\int_T \frac{dz}{z - i} = 2\pi i,$$

where the integral goes once counterclockwise around the boundary of T.

SOLUTION: This follows from the theorem on p. 164, since T is a simple closed contour, and $f = 1$ is analytic everywhere. If you wish to start with the easier Cauchy Integral Formula on circular paths, then pick a circle C of radius r centered at i, where r is so small that the circle is contained in the interior of T. If the integral were on C instead of on (the boundary of) T, then the given equality would be true by the Cauchy Integral Formula for circular paths applied to the constant function 1. Since the integrand is analytic everywhere outside C, deformation of paths shows that the equality is also valid when one integrates over T instead of over C. Yet another way to do this for a circular path is to plug in the parameterization $i + \exp it$ for z, thus reducing to an integral of the constant i from $t = 0$ to $t = 2\pi$.

(4) Let C denote a circle of radius 1 centered at i. Evaluate the integral

$$\int_C \frac{dz}{(z^2+1)^2},$$

where the integral goes once counterclockwise around C. Justify.
Hint: Factor $z^2 + 1$. The answer is a number between 1 and 2.

SOLUTION: Apply the Cauchy Integral Formula for the derivative of f, where $f(z) = (z+i)^{-2}$. This shows that the integral equals $2\pi i f'(i) = \pi/2$.

(5) Given a continuous function $f : \mathbb{C} \to \mathbb{C}$, suppose that for any $z \in \mathbb{C}$,

$$F(z) := \int_0^z f(u)du$$

is a path-independent integral. Using the limit definition of derivative, prove that $F'(z) = f(z)$.

SOLUTION: See proof on page 148.