# Keeler's Theorem and Products of Distinct Transpositions

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**Abstract.** An episode of the television series *Futurama* features a two-body mind-switching machine, which will not work more than once on the same pair of bodies. After the Futurama community engages in a mind-switching spree, the question is asked, "Can the switching be undone so as to restore all minds to their original bodies?" Ken Keeler found an algorithm that undoes any mind-scrambling permutation with the aid of two "outsiders." We refine Keeler's result by providing a more efficient algorithm that uses the smallest possible number of switches. We also present best possible algorithms for undoing two natural sequences of switches, each sequence effecting a cyclic mind-scrambling permutation in the symmetric group $S_n$. Finally, we give necessary and sufficient conditions on $m$ and $n$ for the identity permutation to be expressible as a product of $m$ distinct transpositions in $S_n$.

**1. INTRODUCTION.** "The Prisoner of Benda" [**13**], an acclaimed episode of the animated television series *Futurama*, features a two-body mind-switching machine. Any pair can enter the machine to swap minds, but there is one serious limitation: The machine will not work more than once on the same pair of bodies.

After the Futurama community indulges in a mind-switching frenzy, the question is raised: "Can the switching be undone so as to restore all minds to their original bodies?" The show provides an answer using what is known in the popular culture as "Keeler's theorem" [**5**]. The theorem is the brainchild of the show's writer Ken Keeler [**8**], who earned a Ph.D. in applied mathematics from Harvard University in 1990 [**10**] before becoming a television writer/producer. For "The Prisoner of Benda," Keeler garnered a 2011 Writers Guild Award [**14**].

The problem of undoing the switching can be modeled in terms of group theory. Represent the bodies involved in the switching frenzy by $\{1, 2, \dots, n\}$. The symmetric group $S_n$ consists of the $n!$ permutations of $\{1, 2, \dots, n\}$. Let $I$ denote the identity permutation. A 2-cycle $(ab)$ is called a transposition; it represents the permutation that switches the minds of bodies $a$ and $b$. The $k$-cycle $(a_1 \dots a_k)$ is the permutation that sends $a_1$'s mind to $a_2$, $a_2$'s mind to $a_3$, ..., and $a_k$'s mind to $a_1$. Following the convention in [**1**], we compute products (i.e., compositions) in $S_n$ from right to left. For example, $(123) = (12)(23) = (13)(12) = (23)(13)$.

The successive swapping of minds during the switching frenzy can be represented by a product $P$ of distinct transpositions in $S_n$. (The transpositions must be distinct due to the limitation of the machine.) In addition to viewing $P$ formally as a product, we can also view $P$ as a permutation. It will be assumed that this permutation is nontrivial; otherwise, nothing needs to be undone. For an example of $P$, suppose that 2 switches minds with 3 and then 2 switches minds with 1; this corresponds to the product $P = (12)(23)$, yielding the mind-scrambling permutation $P = (123)$.

To restore all minds to their original bodies, we must find a product $\sigma$ of distinct transpositions such that the permutation $\sigma P$ equals $I$, and such that the transposition factors in the product $\sigma$ are distinct from those in the product $P$. Such a $\sigma$ is said to

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*undo* $P$. From now on, the phrase "transposition factors" will be shortened simply to "factors".

In the aftermath of a switching frenzy, the community may have no recollection of the sequence of switches that had taken place. It is then expedient to find a product $\sigma$ that is guaranteed to undo the mind-scrambling permutation $P \in S_n$ *regardless* of which sequence of transpositions in $S_n$ had effected $P$. Keeler's theorem explicitly produces such a product $\sigma \in S_{n+2}$. Each factor in Keeler's $\sigma$ contains at least one entry in the set $\{x, y\}$, where

$$x := n + 1 \quad \text{and} \quad y := n + 2;$$

hence the factors in $\sigma$ are distinct from whatever transpositions had effected $P$. We can view $x$ and $y$ as altruistic *outsiders* who had never entered the machine during the frenzy, but who are subsequently willing to endure frequent mind switches in order to help others restore their minds to their original bodies.

Viewed as a permutation, $P$ can be expressed (uniquely up to ordering) as the product $P = C_1 \cdots C_r$ of nontrivial disjoint cycles $C_1, \ldots, C_r$ in $S_n$ [**1**, p. 77]. For each $i = 1, \ldots, r$, let $k_i$ denote the length of cycle $C_i$. While discussing Keeler's theorem and our refinement (Theorem 1), we will assume that $k_1 + \cdots + k_r = n$. This presents no loss of generality, since if $k_1 + \cdots + k_r = m < n$, then we could relabel the bodies and mimic the arguments using $m$ in place of $n$.

We now describe Keeler's method for constructing a product $\sigma \in S_{n+2}$ that undoes $P = C_1 \cdots C_r$. For convenience of notation, write $k = k_1$, so that $C_1$ is a $k$-cycle $(a_1 \ldots a_k)$ with each $a_i \in \{1, 2, \ldots, n\}$. It is easily checked that $\sigma_1 C_1 = (xy)$, where $\sigma_1$ is the product of $k + 2$ transpositions given by

$$\sigma_1 = (xa_1)(xa_2) \cdots (xa_{k-1}) \cdot (ya_k)(xa_k)(ya_1). \tag{1}$$

For each $C_i$, define analogous products $\sigma_i$ of $k_i + 2$ transpositions satisfying

$$\sigma_i C_i = (xy) \quad \text{for} \quad i = 1, \ldots, r.$$

Note that every factor of $\sigma_i$ has the form $(xu)$ or $(yu)$ for some entry $u$ in $C_i$. Since disjoint cycles commute, $(xy)$ commutes with every transposition in $S_n$, so $\tau := \sigma_r \cdots \sigma_2 \sigma_1$ is a product of distinct transpositions for which $\tau P = (xy)^r$. Taking

$$\sigma = \begin{cases} (xy)\tau, & \text{if } r \text{ is odd} \\ \tau, & \text{if } r \text{ is even,} \end{cases} \tag{2}$$

we find that $\sigma$ undoes $P$ and $\sigma$ is a product of distinct transpositions in $S_{n+2}$, each containing at least one entry in $\{x, y\}$, as desired.

By (1) and (2), the number of factors in Keeler's $\sigma$ is either $n + 2r + 1$ or $n + 2r$, according to whether $r$ is odd or even. In Theorem 1 of the next section, we refine Keeler's method by showing that $P$ can be undone via a product of only $n + r + 2$ distinct transpositions, each containing at least one entry in $\{x, y\}$. We show, moreover, that this result is best possible in the sense that $n + r + 2$ cannot be replaced by a smaller number. Thus, Keeler's algorithm is optimal for $r = 1$ and $r = 2$, but for no other $r$.

With the aim of finding interesting classes of products that can be undone using *fewer* than two outsiders, we examined what are undoubtedly the two most natural products $P$ in $S_n$ effecting the cycle $(12\ldots n)$, namely [**1**, p. 81]

$$P_1 = (12)(23)(34)\cdots(n-1,n) \quad \text{and} \quad P_2 = (n-1,n)\cdots(3n)(2n)(1n).$$

Theorems 2 and 3 determine how many outsiders and how many mind switches are necessary and sufficient to undo each of these two products. Theorem 2 shows that for $n \geq 5$, $P_1$ can be undone without any outsiders, using only $n+1$ switches, where $n+1$ is best possible. Theorem 3 shows that for $n \geq 3$, $P_2$ can be undone using only one outsider, again with $n+1$ switches, where $n+1$ is best possible.

Suppose for the moment that $n \geq 5$. While $P_1$ and $P_2$ can both be undone with fewer than two outsiders, there are other products $P_3(n)$ in $S_n$ effecting $(12\ldots n)$ for which *two* outsiders are required to undo $P_3(n)$. For an example with $n=5$, let

$$P_3(5) := (54)(53)(52)(51)(12)(23)(14)(13)(24)(34) = (12345).$$

Note that all ten transpositions in $S_5$ are factors of $P_3(5)$. Suppose, for the purpose of contradiction, that $P_3(5)$ can be undone by a product $\sigma$ in $S_6$, i.e., with just one outsider. Every entry in $P_3(5)$ must appear in $\sigma$, so $\sigma$ must be a product of the five factors (61), (62), (63), (64), (65) in some order. The permutation $\sigma$ thus fails to fix the entry 6, which yields the contradiction $\sigma P_3(5) \neq I$. The argument for $n=5$ works the same way for all $n \geq 5$ of the form $4k+1$ or $4k+2$. Simply take $P_3(n) := P_2 J$, where $P_2$ is defined in Theorem 3, and $J$ is the identity formulated as a product of all $\binom{n-1}{2}$ transpositions in $S_{n-1}$, as in Theorem 4. We omit the argument for $n$ of the form $4k$ or $4k+3$, as it's a bit more involved.

The products $P_1$ and $P_2$ each have the property that no two consecutive factors are disjoint. In contrast, consider the product of $m$ disjoint factors

$$P(m) := (12)(34)\ldots(2m-1,2m).$$

We call $P(m)$ the *Stargate switch* because $P(2)$ represents a sequence of mind swaps featured in an episode of the sci-fi television series *Stargate SG-1* [**4**]. The first and second authors [**3**] have given an optimal algorithm for undoing $P(m)$; for $m > 1$, the algorithm requires no outsiders.

When $n \geq 5$, Theorem 2 provides equalities of the form $\sigma P_1 = I$, which express the identity $I$ as a product of $2n$ distinct transpositions in $S_n$. Such equalities lead to the question: What are necessary and sufficient conditions on $m$ and $n$ for $I$ to be expressible as a product of $m$ distinct transpositions in $S_n$? Theorem 4 provides the answer. It is necessary and sufficient that $m$ be an even integer with $6 \leq m \leq \binom{n}{2}$.

In order to prove Theorems 2–4, we require some properties of cycles proved via graph theory in Lemma 1. The proof of Lemma 1(c) incorporates an idea of Jacques Verstraete in a proof due to Isaacs [**7**]. We are grateful for their permission to include it here, as our original proof was considerably less elegant.

We will also need the well-known "Parity theorem," which shows that the identity permutation $I$ cannot equal a product of an odd number of transpositions. Two proofs of the Parity theorem may be found in [**1**, pp. 82, 149]; for an elegant recent proof, see Oliver [**12**].

**2. AN OPTIMAL REFINEMENT OF KEELER'S METHOD.** Keeler's algorithm was designed to undo every mind-scrambling permutation $P = C_1 \cdots C_r$ that is effected by an unknown sequence of mind swaps. In this section, we present another such algorithm. While Keeler's algorithm is optimal only for $r \leq 2$, we prove that our algorithm is optimal for all $r$.

**Theorem 1.** *Let $P = C_1 \cdots C_r$ be a product of $r$ disjoint $k_i$-cycles $C_i$ in $S_n$, with $k_i \geq 2$ and $n = k_1 + \cdots + k_r$. Define $x = n + 1$ and $y = n + 2$. Then $P$ can be undone by a product $\lambda$ of $n + r + 2$ distinct transpositions in $S_{n+2}$, each containing at least one entry in $\{x, y\}$. Moreover, this result is best possible in the sense that $n + r + 2$ cannot be replaced by a smaller number.*

*Proof.* Write $k = k_1$, so that $C_1$ is a $k$-cycle $(a_1 \ldots a_k)$. Corresponding to the cycle $C_1$, define

$$G_1(x) = (a_1x)(a_2x)\cdots(a_kx) \quad \text{and} \quad F_1(x) = (a_1x).$$

Corresponding to each cycle $C_i$ for $i = 1, \ldots, r$, define $G_i(x)$ and $F_i(x)$ analogously. Set

$$\lambda = (xy) \cdot G_r(x) \cdots G_2(x) \cdot (a_kx)G_1(y)(a_1x) \cdot F_2(y) \cdots F_r(y).$$

It is readily checked that $\lambda$ undoes $P$ and that $\lambda$ is a product of $n + r + 2$ distinct transpositions in $S_{n+2}$, each containing at least one entry in $\{x, y\}$.

It remains to prove optimality. Suppose, for the purpose of contradiction, that $P$ can be undone by a product $\sigma$ of $t < n + r + 2$ distinct transpositions in $S_{n+2}$, each containing at least one entry in $\{x, y\}$. Then by the Parity theorem, $t \leq n + r$.

On the other hand, we have the lower bound $t \geq n$, since each of the $n$ entries in $P$ must occur (coupled with $x$ or $y$) in a factor of $\sigma$. Let $A$ denote the set of entries in $C_1 = (a_1 \ldots a_k)$, and let $a$ denote the rightmost element of $A$ appearing in the product $\sigma$. Since $P$ maps $a$ to some other element of $A$, it follows that $a$ appears twice in $\sigma$, i.e., $\sigma$ has both of the factors $(ax)$ and $(ay)$. The same argument shows that each of the $r$ cycles $C_i$ contains an entry that appears twice in $\sigma$. Thus, the inequality $t \geq n$ can be strengthened to $t \geq n + r$. Consequently, $t = n + r$. It follows that each of the $r$ cycles $C_i$ contains exactly one entry that appears twice in $\sigma$, and the other $n - r$ entries appear only once. This accounts for all $n + r$ factors of $\sigma$, so in particular, $(xy)$ cannot be a factor of $\sigma$.

Let $a'$ denote the leftmost element of $A$ appearing in the product $\sigma$. Since $P$ maps some element of $A$ to $a'$, it follows that $a'$ appears twice in $\sigma$. Since $a$ is the only element of $A$ that appears twice in $\sigma$, we must have $a = a'$. Consequently, we have shown the following two properties of $C_1$:

(i) there is a unique entry $a$ in $C_1$ for which the transpositions $(ax)$ and $(ay)$ both occur as factors of $\sigma$, and

(ii) each entry of $C_1$ other than $a$ occurs in exactly one factor of $\sigma$, and that factor lies strictly between $(ax)$ and $(ay)$.

These two properties are similarly shared by each of the $r$ cycles $C_i$.

Let $N_1$ denote the number of transpositions in $\sigma$ that lie strictly between its factors $(ax)$ and $(ay)$. Define $N_i$ similarly for each of the $r$ cycles $C_i$. We may assume without loss of generality that $N_1 \leq N_i$ for all $i$. We may also assume that the factor $(ax)$ in $\sigma$ lies to the left of the factor $(ay)$, and that $a = a_k$.

Let $M_y$ denote the set of factors in $\sigma$ that contain the entry $y$ and that lie between $(a_kx)$ and $(a_ky)$ inclusive. Suppose, for the purpose of contradiction, that every transposition in $M_y$ has the form $(a_iy)$ for some $a_i \in A$. Since $\sigma$ must send $a_{i+1}$ to $a_i$ for each $i = 1, \ldots, k - 1$, it follows that the elements of $M_y$ have to occur in the following order in $\sigma$:

$$(a_1y), (a_2y), \ldots, (a_{k-1}y), (a_ky).$$

But then $\sigma$ could not send $a_1$ to $a_k$, a contradiction. Thus some transposition in $M_y$ must have the form $(hy)$, where $h \notin A$. Consider the rightmost $(hy) \in M_y$ with $h \notin A$. For some fixed $j > 1$, $h$ is an entry of the cycle $C_j$. Among all the elements $(a_i y) \in M_y$ that lie to the right of $(hy)$, let $(a_m y)$ denote the one closest to $(hy)$. As $\sigma$ cannot send $a_m$ to $h$, it follows that the entry $h$ occurs twice between $(a_k x)$ and $(a_k y)$, i.e., $\sigma$ has factors $(hx)$ and $(hy)$ both lying strictly between $(a_k x)$ and $(a_k y)$. Thus, $N_j < N_1$. This violates the minimality of $N_1$, giving us the desired contradiction. ∎

## 3. A LEMMA ON FACTORIZATIONS OF CYCLES.

**Lemma 1.** *For $2 \leq k \leq n$, suppose that the $k$-cycle $(a_1 \dots a_k) \in S_n$ equals a product $P$ of $t$ transpositions in $S_n$. Then*

(a) $t \geq k - 1$,

(b) *when $t = k - 1$, the set of entries in $P$ is $V := \{a_1, \dots, a_k\}$, and*

(c) *when $t = k - 1$, at least one factor of $P$ has the form $(a_i a_{i+1})$ with $1 \leq i < k$.*

*Proof.* Since $(ij)(ab)(ij)$ equals a transposition, a product of nondistinct transpositions reduces to a shorter product of distinct transpositions. Thus, it suffices to prove the result when the factors of $P$ are distinct. Let $W$ denote the set of entries in the product $P$. Note that $W$ contains the set $V := \{a_1 \dots a_k\}$. Define a graph $G$ with vertex set $W$ and with $t$ edges $[i, j]$ corresponding to the $t$ transposition factors $(ij)$ of $P$. Since $P = (a_1 \dots a_k)$ is a product of these $t$ transpositions, the graph $G$ has a connected component $H$ whose vertex set contains $V$. A connected graph with $M$ vertices has at least $M - 1$ edges [**2**, Theorem 11.2.1, p. 163], so $H$ and hence $G$ must have at least $|V| - 1 = k - 1$ edges. Thus $t \geq k - 1$. This proves part (a). (For another proof of part (a), see [**6**, p. 77]. For a generalization proved via linear algebra, see [**9**].)

For the rest of this proof, suppose that $t = k - 1$. Then $H$ has $t$ edges, so $G = H$ and $G$ is connected. If $V$ were *strictly* contained in $W$, then again by [**2**, Theorem 11.2.1, p. 163], $G$ would have at least $k$ edges. Thus $V = W$, which proves part (b). (For a generalization of part (b), see [**11**].)

To prove part (c), it remains to prove that one of the $k - 1$ edges of $G$ has the form $[a_i, a_{i+1}]$ with $1 \leq i < k$. This is clear for $k = 2$, so we let $k \geq 3$ and induct on $k$. A connected graph with $k$ vertices is a tree if and only if it has $k - 1$ edges [**2**, Theorem 11.2.1, p. 163]. Thus $G$ is a tree. Let $(a_u a_v)$ denote the rightmost factor of $P$, with $u < v$. Write $w = v - u$. If $w = 1$, we are done, so assume that $w > 1$. Define the disjoint cycles

$$r = (a_{u+1} \dots a_v) \quad \text{and} \quad s = (a_1 \dots a_u, a_{v+1}, \dots, a_k),$$

so that $r$ is a $w$-cycle and $s$ is a $(k - w)$-cycle. If $v = k$, then $s$ is interpreted as $(a_1 \dots a_u)$, which in turn is interpreted as the identity permutation when $u = 1$. Define $P'$ to be the product obtained from $P$ by removing the rightmost factor $(a_u a_v)$. Let $G'$ be the graph obtained from $G$ by removing the edge $[a_u, a_v]$. Then $P'$ has $k - 2$ factors and $G'$ has $k - 2$ edges. Since $P = sr(a_u a_v)$, we have $P' = sr$. It follows that $G'$ is a forest of two trees $R$ and $S$, where $R$ is a tree on the $w$ vertices $a_{u+1}, \dots, a_v$, and $S$ is a tree on the remaining vertices in $V$. The $w$-cycle $r$ equals a product $Q$ of the $w - 1$ factors of $P$ corresponding to the $w - 1$ edges of $R$. Since $w < k$, it follows by induction that $Q$, and hence $P$, has a factor of the required form $(a_i a_{i+1})$. ∎

## 4. OPTIMAL METHODS TO UNDO $P_1$ AND $P_2$.

**Theorem 2.** *For $n \geq 5$, let $P_1$ denote the product of $n-1$ transpositions in $S_n$ given by $P_1 = (12)(23)(34)\cdots(n-1,n)$. There exists a product $\sigma$ of $n+1$ distinct transpositions in $S_n$ that undoes $P_1$, and this result is best possible in the sense that no such $\sigma$ can have fewer than $n+1$ distinct factors.*

*Proof.* Define

$$\sigma = (3n)(2, n-1)(1n)(14)(2n)(13)\cdot(35)\cdots(3, n-1),$$

where, when $n = 5$, the empty product $(35)\cdots(3, n-1)$ is interpreted as the identity. It is easily checked that $\sigma P_1 = I$ and that $\sigma$ is a product of $n+1$ distinct transpositions in $S_n$, all distinct from the $n-1$ transpositions in $P_1$. It remains to prove optimality.

Suppose, for the purpose of contradiction, that there exists a product $E$ of $k < n+1$ distinct transpositions in $S_n$ for which $EP_1 = I$ and for which the $k$ transpositions in $E$ are distinct from the $n-1$ transpositions in $P_1$. Since $EP_1 = I$, the Parity theorem shows that $k \leq n-1$. On the other hand, since $P_1 = (12\ldots n)$, Lemma 1(a) gives $k \geq n-1$. Thus, the number of transpositions in the product $E$ is exactly $n-1$. Note that $E^{-1}$ is a product of these same $n-1$ transpositions in reverse order, and $E^{-1} = P_1 = (12\ldots n)$. Hence by Lemma 1(c), one of these $n-1$ transpositions in $E$ has the form $(i, i+1)$ with $1 \leq i < n$. This contradicts the distinctness of the factors of $E$ from those in $P_1$, since by definition $P_1$ is a product of all $n-1$ transpositions $(i, i+1)$ with $1 \leq i < n$. ■

**Theorem 3.** *For $n \geq 3$, let $P_2$ denote the product of $n-1$ transpositions in $S_n$ given by $P_2 = (n, n-1)\cdots(n3)(n2)(n1)$. There exists a product $\tau$ of $n+1$ distinct transpositions in $S_{n+1}$ that undoes $P_2$, and this result is best possible in the sense that no such $\tau$ can have fewer than $n+1$ distinct factors.*

*Proof.* Define

$$\tau = (2, n+1)(3, n+1)(4, n+1)\cdots(n, n+1)\cdot(1,2)(1, n+1).$$

It is easily checked that $\tau P_2 = I$ and that $\tau$ is a product of $n+1$ distinct transpositions in $S_{n+1}$, all distinct from the $n-1$ transpositions in $P_2$. It remains to prove optimality.

Suppose, for the purpose of contradiction, that there exists a product $F$ of $k < n+1$ transpositions in $S_{n+1}$ for which $FP_2 = I$ and for which the $k$ transpositions in $F$ are distinct from the $n-1$ transpositions in $P_2$. Since $FP_2 = I$, the Parity theorem shows that $k \leq n-1$. On the other hand, since $P_2 = (1, 2\ldots n)$, Lemma 1(a) gives $k \geq n-1$. Thus, the number of transpositions in the product $F$ is exactly $n-1$. Note that $F^{-1}$ is a product of these same $n-1$ transpositions in reverse order, and $F^{-1} = P_2 = (1, 2, \ldots, n)$. Hence, by Lemma 1(b), the entries in these $n-1$ transpositions all lie in the set $\{1, 2\ldots n\}$. Since the permutation $F$ moves $n$, it follows that one of these $n-1$ transpositions in $F$ has the form $(in)$ with $1 \leq i < n$. This contradicts the distinctness of the factors of $F$ from those in $P_2$, since by definition, $P_2$ is a product of all $n-1$ transpositions $(in)$ with $1 \leq i < n$. ■

**Remark.** When $n = 2$, two outsiders are required to undo $P_1 = P_2 = (12)$, and an optimal $\sigma$ is given by $(34)(23)(14)(24)(13)$. In the cases $n = 3$ and $n = 4$, one outsider is required to undo $P_1$, and optimal $\sigma$'s are given by $(14)(13)(24)(34)$ and $(14)(25)(24)(35)(45)$, respectively.

## 5. $I$ AS A PRODUCT OF $m$ DISTINCT TRANSPOSITIONS IN $S_n$.

**Theorem 4.** *For the identity $I$ to be expressible as a product of $m$ distinct transpositions in $S_n$, it is necessary and sufficient that $m$ be an even integer with $6 \le m \le \binom{n}{2}$.*

*Proof.* We begin by showing that the conditions are necessary. First, $m$ must be even by the Parity theorem, and it is not hard to show that $m$ cannot equal 2 or 4. Furthermore, $m$ cannot exceed $\binom{n}{2}$, since $\binom{n}{2}$ is the number of distinct transpositions in $S_n$. This proves necessity, and it remains to show sufficiency.

Define $f(a, b, c) = (ac)(ab)(bc)$, which we view formally as a product of three transpositions, while noting that $f(a, b, c)$ equals $(ab)$ when viewed as a permutation. If a product $\lambda$ of transpositions has a factor $(ab)$, then formally replacing $(ab)$ by $f(a, b, c)$ increases the number of $\lambda$'s factors by 2, without altering $\lambda$ as a permutation.

For even $m$ in the appropriate range, we now show how to express $I$ explicitly as a product of $m$ distinct transpositions in $S_4$, $S_5$, $S_6$, $S_7$, and $S_8$. An analogous treatment will then inductively express $I$ as a product of $m$ distinct transpositions in $S_{4k}$, $S_{4k+1}$, $S_{4k+2}$, $S_{4k+3}$, and $S_{4k+4}$ for all $k \ge 2$, thus completing the proof.

For $m = 6$, we have the base case

$$I = (12)(23)(14)(13)(24)(34) \quad \text{in } S_4.$$

This equality uses all six transpositions in $S_4$, so to consider the values $m = 8, 10$, we move up to $S_5$. For $m = 8$, replace the first transposition (12) above by $f(1, 2, 5)$ to obtain

$$I = (15)(12)(25)(23)(14)(13)(24)(34) \quad \text{in } S_5.$$

For $m = 10$, replace the transposition (34) above by $f(3, 4, 5)$ to obtain

$$I = (15)(12)(25)(23)(14)(13)(24)(35)(34)(45) \quad \text{in } S_5.$$

This equality uses all ten transpositions in $S_5$, so to consider the values $m = 12, 14$, we move up to $S_6$. For $m = 12$, replace (23) above by $f(2, 3, 6)$ to obtain

$$I = (15)(12)(25)(26)(23)(36)(14)(13)(24)(35)(34)(45) \quad \text{in } S_6.$$

For $m = 14$, replace (45) above by $f(4, 5, 6)$ to obtain

$$I = (15)(12)(25)(26)(23)(36)(14)(13)(24)(35)(34)(46)(45)(56) \quad \text{in } S_6.$$

This equality uses all of the fifteen transpositions in $S_6$ except for (16), so to consider the values $m = 16, 18, 20$, we move up to $S_7$. For $m = 16$, $m = 18$, and $m = 20$, successively replace (12) by $f(1, 2, 7)$, (34) by $f(3, 4, 7)$, and (56) by $f(5, 6, 7)$, respectively. This yields the following for $m = 20$:

$$I = (15)(17)(12)(27)(25)(26)(23)(36)(14)(13)(24)(35) \times (37)(34)(47)(46)(45)(57)(56)(67) \quad \text{in } S_7.$$

This equality uses all of the twenty-one transpositions in $S_7$ except for (16), so to consider the values $m = 22, 24, 26, 28$, we move up to $S_8$. For $m = 22$, $m = 24$, and $m = 26$, successively replace (23) by $f(2, 3, 8)$, (45) by $f(4, 5, 8)$, and (67) by $f(6, 7, 8)$, respectively. This yields the following for $m = 26$:

$$I = (15)(17)(12)(27)(25)(26)(28)(23)(38)(36)(14)(13)(24)(35)(37)$$
$$\times (34)(47)(46)(48)(45)(58)(57)(56)(68)(67)(78) \quad \text{in } S_8.$$

This equality uses all twenty-eight transpositions in $S_8$ except (16) and (18). This suggests that we make the atypical replacement of (68) by $f(6, 8, 1)$ to obtain the following for $m = 28$:

$$I = (15)(17)(12)(27)(25)(26)(28)(23)(38)(36)(14)(13)(24)(35)(37)$$
$$\times (34)(47)(46)(48)(45)(58)(57)(56)(16)(68)(18)(67)(78) \quad \text{in } S_8.$$

This equality uses all twenty-eight transpositions in $S_8$. From here, we can repeat the procedure. ■

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## Non-recursive Polynomial Formula for the Sum of the Powers of the Integers

There are *recursive* formulas for the sum $S_n$ of the $p$th powers of the first $n$ positive integers. Here we present a *non-recursive* polynomial formula $P(n)$.

**Theorem.** *If* $S_n = 1^p + \cdots + n^p$ *(where* $n, p \in \mathbb{N}_0$, *and* $S_0 = 0$*), then* $S_n = P(n)$, *where* $P(n)$ *is the polynomial* $P(n) = \sum_{i=1}^{p+1} S_i Q_i(n)/Q_i(i)$ *of degree* $\deg P(n) = p + 1$ *and* $Q_i(n) = \prod_{\substack{j=0 \\ j \neq i}}^{p+1} (n - j)$.

*Proof.* We have $S_n = P(n)$ for $n = 0, \ldots, p + 1$ by definition of $P(n)$ and $Q_i(n)$. So $P(n) - P(n-1) = n^p$ for the $p + 1$ values $n = 1, \ldots, p + 1$. But $\deg(P(n) - P(n-1)), \deg n^p < p + 1$; thus $(*)$ $P(n) - P(n-1) = n^p$. Therefore, $S_n = P(n)$ by induction on $n$. We have $\deg P(n) = p + 1$, otherwise $\deg(P(n) - P(n-1)) < p$, contradicting $(*)$. ■

**Example.** We have

$$1^2 + \cdots + n^2 = \left\{ \begin{array}{r} 1^2 \dfrac{(n-0)(n-2)(n-3)}{(1-0)(1-2)(1-3)} + \\ (1^2+2^2) \dfrac{(n-0)(n-1)(n-3)}{(2-0)(2-1)(2-3)} + \\ (1^2+2^2+3^2) \dfrac{(n-0)(n-1)(n-2)}{(3-0)(3-1)(3-2)} \end{array} \right\} = \frac{n(n+1)(2n+1)}{6}.$$

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