PFAFFIANS AND STRATEGIES FOR INTEGER CHOICE GAMES

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To Roger Howe on his sixtieth birthday.

Abstract. The purpose of this paper is to develop optimal strategies for a simple integer choice game with a skew symmetric payoff matrix. The analysis involves the calculation of certain Pfaffians associated with these matrices.

1. Introduction

Alice and Bob play a game where each secretly chooses a positive integer. If both players choose the same integer then the game is a tie. Otherwise, the player that chooses the smallest integer (say Bob) wins \$1 (from Alice), unless the two integers differ by 1, in which case Alice wins w dollars from Bob. Here w is an arbitrary positive integer. This game was partially analyzed by Mendelsohn [4] about 60 years ago. It was also discussed in the book of Herstein and Kaplansky [3] and was further popularized in a book of Martin Gardner [2, Chapter 9.3].

It will be convenient to use "negative payoffs". For example, if Bob's integer is 1 less than Alice's, then Bob receives the negative payoff $v = -w$. That means that Bob pays Alice w dollars. This game yields a skew-symmetric payoff matrix $A(v)$ for "the row player" Bob, with v on the first super-diagonal and all 1's above. We denote by $A_m(v)$ the $m \times m$ submatrix of $A(v)$ consisting of the first m rows and columns; for example,

$$A_5(v) = \begin{bmatrix} 0 & v & 1 & 1 & 1 \\ -v & 0 & v & 1 & 1 \\ -1 & -v & 0 & v & 1 \\ -1 & -1 & -v & 0 & v \\ -1 & -1 & -1 & -v & 0 \end{bmatrix}.$$

If both players choose integers from the set $\{1, \ldots, m\}$ then $A_m(v)$ is the payoff matrix for the row player Bob. For example, if Bob plays 3 and Alice plays 1, 2, 3, 4 or 5, then Bob wins -1, $-v$, 0, v or 1, respectively.

A *strategy* for a player of this game is a list of plays each with a corresponding probability. For example, Bob could have the strategy of playing 1, 2, and 4 with probabilities 2/3, 1/4, and 1/12, respectively (playing every other integer with probability 0). A *pure strategy* is a strategy where one probability is 1 and all of the rest are 0. For example, Bob's strategy is pure if he plays 4 with probability 1.

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An *optimal strategy* is a strategy that beats or ties any other strategy. It is easily seen that a strategy is optimal if and only if it beats or ties every pure strategy.

Mendelsohn [4] found numbers $v_4 < v_3 < v_2 < v_1 = 0$ (given below) such that: if $v_2 < v < v_1$ then the unique optimal strategy entails playing only the integers $1, 2, 3$; if $v_3 < v < v_2$ then the unique optimal strategy plays only the integers $1, 2, 3, 4, 5$; and if $v_4 < v < v_3$ then the unique optimal strategy plays only the integers $1, 2, 3, 4, 5, 6, 7$. For example, $v_2 = (-1 - \sqrt{5})/2$ and when $v_2 < v < 0$, the unique optimal strategy is to randomly choose the integers $1, 2, 3$ with respective probabilities $v/(2v - 1)$, $1/(1 - 2v)$, $v/(2v - 1)$. The aspect of this game that we find to be most striking (and our main reason for studying it) is the radical change in strategy that can be caused by a small change in the payoff v. In Theorem 1.1 below, we extend Mendelsohn's results by explicitly determining the unique optimal strategies for all $v < 0$ for which they exist. An equivalent (but less elegant) version of Theorem 1.1 was stated without proof in [1, Appendix].

For $m \geq 1$, set

$$v_m = 1 - \frac{1}{2(1 - \cos \frac{\pi}{2m+1})}.$$

This is consistent with the notation v_1, v_2, v_3, v_4 in the last paragraph. We have

$$0 = v_1 > v_2 > v_3 > v_4 > \dots$$

with

$$v_m = -\left(\frac{2m+1}{\pi}\right)^2 + O(1),$$

so v_m tends to $-\infty$ quadratically.

We define a sequence of polynomials $F_m = F_m(x)$ recursively as follows: $F_{-1} = 0$, $F_0 = 1$ and

$$F_{m+1} = F_m + (x - 1)F_{m-1}, \quad m \geq 0.$$

The polynomials indexed by 1, 2, 3, 4, 5, 6 are respectively

$$1,\ x,\ 2x - 1,\ x^2 + x - 1,\ 3x^2 - 2x,\ x^3 + 3x^2 - 4x + 1.$$

For $1 \leq j \leq m$ we define rational functions in v by the formula

$$p_j(m) = (-1)^{j+1} \frac{F_{j-1}(v)F_{m-j}(v)}{F_m(v)}.$$

The advertised result, which is the main content of Theorem 3.4, is

Theorem 1.1. *Suppose that there is an odd number k such that*

$$v_{k+1} < v < v_k.$$

Then $p_j(2k+1) > 0$ for $j = 1, \dots, 2k+1$, $\sum_{j=1}^{2k+1} p_j(2k+1) = 1$, and the unique optimal strategy is to choose $1,\ 2, \dots, 2k+1$ with respective probabilities

$$p_1(2k+1),\ p_2(2k+1), \dots, p_{2k+1}(2k+1).$$

This integer choice game can be generalized so that for each $i \geq 1$, whenever Bob chooses i and Alice chooses $i + 1$, Bob's payoff is an amount $x_i < 0$ in lieu of the constant amount v. We will call this more general version the "multivariate game".

The multivariate analogue of the payoff matrix $A_n(v)$ is the skew-symmetric $n \times n$ matrix

$$A_n = A_n(x_1, \dots, x_{n-1}) = \begin{bmatrix} 0 & x_1 & 1 & 1 & \dots & 1 & 1 \\ -x_1 & 0 & x_2 & 1 & \dots & 1 & 1 \\ -1 & -x_2 & 0 & x_3 & \dots & 1 & 1 \\ -1 & -1 & -x_3 & 0 & \dots & 1 & 1 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ -1 & -1 & -1 & -1 & \dots & 0 & x_{n-1} \\ -1 & -1 & -1 & -1 & \dots & -x_{n-1} & 0 \end{bmatrix}.$$

For indeterminates $x_0, x_1, \dots,$ we can define multivariate analogues $F_n = F_n(x_1, \dots, x_{n-1})$ of the polynomials $F_n(x)$ recursively as follows: $F_{-1} = 0$, $F_0 = 1$ and

$$F_{m+1} = F_m + (x_m - 1)F_{m-1}, \quad m \geq 0.$$

These polynomials are intimately related to the payoff matrices A_n. For example, it will be seen in Section 5 that if n is even, $F_n(x_1, \dots, x_{n-1})$ is the Pfaffian of $A_n(x_1, \dots, x_{n-1})$.

Let $\mathbf{x}$ denote the infinite vector $(x_1, x_2, \dots)$. In contrast with the single variable case, we do not know an explicit characterization of the set of all vectors $\mathbf{x}$ such that the multivariate game has a unique optimal strategy. However, in Theorem 4.2 we present the unique optimal strategy for the multivariate game in the special case that the first $2m+1$ entries of $\mathbf{x}$ are contrained to the interior of an explicitly given unit hypercube.

We collect together results on the polynomials $F_n = F_n(x_1, \dots, x_{n-1})$ and the payoff matrices $A_n = A_n(x_1, \dots, x_{n-1})$ in the Appendix (Section 5). The main result in the Appendix, Theorem 5.8, shows that if no x_i equals 1, then $\ker A_{2k+1}(x_1, \dots, x_{2k})$ is a one-dimensional space $\mathbb{R}\mathbf{t}$ where $\mathbf{t}$ is explicitly expressed in terms of the polynomials F_n as well as in terms of Pfaffians of the diagonal minors of A_{2k+1}. Theorem 5.8 is instrumental in the proof of Proposition 2.2. Propositions 2.1 and 2.2 in Section 2 give a general analysis of optimal strategies for the multivariate game. Proposition 2.2 is applied to prove the main theorems in Sections 3 and 4 (Theorems 3.4 and 4.2).

As indicated above, this paper is dedicated to Roger Howe. We hope that he enjoys it as much as we enjoyed writing it.

2. Strategies for the multivariate game

In this section, we provide methods for constructing optimal strategies for the multivariate game described in Section 1.

Proposition 2.1. *Suppose that there exists a vector* $\mathbf{p} = (p_1, \dots, p_{2k+1}) \in \mathbb{R}^{2k+1}$ *such that*

$$\ker A_{2k+1}(x_1, \dots, x_{2k}) = \mathbb{R}\mathbf{p}, \tag{2.1}$$

$$\textstyle\sum_{i=1}^{2k+1} p_i = 1, \quad \textit{with all } p_i > 0, \tag{2.2}$$

and

$$\sum_{i=1}^{2k} p_i + x_{2k+1} p_{2k+1} > 0. \tag{2.3}$$

Then the unique optimal strategy is to play i with probability p_i, for $1 \le i \le 2k+1$. This strategy is still optimal (but not necessarily unique) if the inequalities in (2.2) *and* (2.3) *are not required to be strict.*

Proof. Let Bob play j with probability p_j for $1 \le j \le 2k+1$. Suppose that Alice plays the pure strategy i. If $i \le 2k+1$, then Bob's payoff is

$$p_1 + \cdots + x_{i-1}p_{i-1} - x_i p_{i+1} - p_{i+2} + \cdots$$

which vanishes by (2.1). If $i > 2k+2$, then Bob's payoff is 1. If $i = 2k+2$, then Bob's payoff is

$$p_1 + \cdots + p_{2k} + x_{2k+1}p_{2k+1} > 0$$

by (2.3). Thus Bob beats or ties every pure strategy, so his strategy is optimal. This argument shows that Bob's strategy is still optimal if the inequalities in (2.2) and (2.3) are not required to be strict.

We now prove uniqueness. Suppose that against Bob's optimal strategy, Alice plays an optimal strategy in which she chooses i with probability r_i for $i \ge 1$. We have seen that Bob beats every pure strategy exceeding $2k+1$, so $r_i = 0$ for every $i > 2k+1$. For brevity, let A denote the payoff matrix in (2.1), and let $\mathbf{r}$ denote the column vector $(r_1, \ldots, r_{2k+1})$. Since Alice's strategy is optimal, all $2k+1$ entries in the vector $A\mathbf{r}$ are ≤ 0. If at least one of these entries were strictly negative, then by (2.2), we would have $\mathbf{p}A\mathbf{r} < 0$. This is impossible, since $\mathbf{p}A = 0$ by (2.1). Thus $A\mathbf{r} = 0$. Hence by (2.1), $\mathbf{r}$ is a scalar multiple of $\mathbf{p}$. Since the sum of the entries of $\mathbf{r}$ and the sum of the entries of $\mathbf{p}$ both equal 1, we have $\mathbf{r} = \mathbf{p}$, which completes the proof of uniqueness. □

We will now apply results in the Appendix to refine Proposition 2.1.

Proposition 2.2. *Assume that $x_i < 0$ for $i \ge 1$ and that*

$$F_{2k+1}(x_1, \ldots, x_{2k}) \ne 0, \tag{2.4}$$

$$p_i := \frac{(-1)^{i+1} F_{i-1}(x_1, \ldots, x_{i-2}) F_{2k+1-i}(x_{i+1}, \ldots, x_{2k})}{F_{2k+1}(x_1, \ldots, x_{2k})} > 0, \quad 1 \le i \le 2k+1, \tag{2.5}$$

and

$$\frac{F_{2k+2}(x_1, \ldots, x_{2k+1})}{F_{2k+1}(x_1, \ldots, x_{2k})} > 0. \tag{2.6}$$

Then $p_1 + \cdots + p_{2k+1} = 1$ and the unique optimal strategy is to play $i = 1, \ldots, 2k+1$ with probabilities $p_1, \ldots, p_{2k+1}$, respectively. This strategy is still optimal if the inequalities in (2.5) *and* (2.6) *are not required to be strict.*

Proof. In the notation of (2.5), write $\mathbf{p} = (p_1, \ldots, p_{2k+1})$. We need only check that the three conditions of Proposition 2.1 hold. By (2.4) and (2.5) and Lemma 5.6, we have $p_i > 0$ and $p_1 + \cdots + p_{2k+1} = 1$. By Theorem 5.8, $\ker A_{2k+1}(x_1, \ldots, x_{2k}) = \mathbb{R}\mathbf{p}$. Thus (2.1) and (2.2) are proved, and it remains to check (2.3). The left side of (2.3) equals

$$\frac{\left(\sum_{i=1}^{2k} (-1)^{i+1} F_{i-1}(x_1, \ldots, x_{i-2}) F_{2k+1-i}(x_{i+1}, \ldots, x_{2k}) \right)}{F_{2k+1}(x_1, \ldots, x_{2k})} + \frac{x_{2k+1} F_{2k}(x_1, \ldots, x_{2k-1})}{F_{2k+1}(x_1, \ldots, x_{2k})} =$$

$$\frac{\sum_{i=1}^{2k+1}(-1)^{i+1}F_{i-1}(x_1,\dots,x_{i-2})F_{2k+1-i}(x_{i+1},\dots,x_{2k})+(x_{2k+1}-1)F_{2k}(x_1,\dots,x_{2k-1})}{F_{2k+1}(x_1,\dots,x_{2k})}$$

and Lemma 5.6 implies that this expression is equal to

$$\frac{F_{2k+1}(x_1,\dots,x_{2k})+(x_{2k+1}-1)F_{2k}(x_1,\dots,x_{2k-1})}{F_{2k+1}(x_1,\dots,x_{2k})}.$$

By Lemma 5.4, this in turn equals the positive expression in (2.6). This completes the proof of (2.3). □

We remark that Proposition 2.2 can also be proved by ad hoc methods which are more elementary (but less elegant).

Consider the strategy of choosing i with probability p_i for $i \geq 1$. We say this strategy is *finite* if $p_j = 0$ for all sufficiently large j. The next result provides an example of a game with an infinite but no finite optimal strategy.

Proposition 2.3. *If $x_i = -(2^{i+1}-3)$ for $i = 1,\ 2, \dots$ then an optimal strategy is to play i with probability 2^{-i} for each $i \geq 1$. This game has no finite optimal strategy.*

Proof. Let $\mathbf{r} = (r_1,\ r_2, \dots)$ with $r_i = 2^{-i}$. The strategy of playing i with probability r_i for $i \geq 1$ ties every pure strategy n, because

$$-\sum_{i\leq n-2}2^{-i}-x_{n-1}2^{1-n}+x_n2^{-n-1}+\sum_{i>n+1}2^{-i}=0.$$

This infinite strategy is thus optimal.

Now consider another optimal strategy in which i is played with probability p_i for $i \geq 1$, where $1 = p_1 + p_2 + \cdots$. Let $\mathbf{p} = (p_1, p_2, \dots)$, viewed as an infinite column vector. For the infinite payoff matrix A, we have $0 = \mathbf{r}A = \mathbf{r}A\mathbf{p}$. Since all entries of $A\mathbf{p}$ are ≤ 0, this implies that $A\mathbf{p} = 0$. Suppose for the purpose of contradiction that $p_i = 0$ for all $i > N$, where without loss of generality, $N = 2m$ is even. Then the submatrix $A_{2m}(x_1, \dots, x_{2m-1})$ of A has a nontrivial kernel, so its determinant and thus its Pfaffian vanishes. As was noted above Lemma 5.4, the Pfaffian of $A_{2m}(x_1, \dots, x_{2m-1})$ is $F_{2m}(x_1, \dots, x_{2m-1})$. One can show using the recurrence that

$$F_{2m}(x_1,\dots,x_{2m-1})=(-1)^m(2-1)(2^3-1)\cdots(2^{2m-1}-1).$$

Since this is nonzero, we have the desired contradiction to the assumption that a finite optimal strategy exists. □

We remark that the optimal strategy given in Proposition 2.3 is not unique. In fact, for any a with $0 \leq a \leq 1/2$, it is optimal to play i with probability p_i for $i \geq 1$, where the sequence p_i is defined by the recurrence $p_1 = a$, $p_2 = (1-a)/2$, $p_3 = (1+a)/12$, and for $n \geq 4$,

$$(2^n-2)p_n=(2^{n-1}-3)p_{n-1}+(2^{n-1}-3)p_{n-2}+(2-2^{n-2})p_{n-3}.$$

For each fixed a, we have $1 = p_1 + p_2 + \cdots$ and $p_i > 0$, except that $p_1 = 0$ in the case that $a = 0$. The case $a = 1/2$ gives the optimal strategy presented in Proposition 2.3.

3. Strategies for the single variable game

In this section we will assume that all $x_i = v < 0$. We will write $A_n(v)$ for $A_n(v, v, \ldots, v)$ and $F_n(v)$ for $F_n(v, v, \ldots, v)$. Our goal is to prove Theorem 3.4.

We have the recurrence relation $F_{-1} = 0,\ F_0 = 1,\ F_1 = 1$ and for $n \geq 0$,

$$F_{n+2}(v) = F_{n+1}(v) + (v-1)F_n(v).$$

Since

$$\begin{bmatrix} 1 & v-1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} F_{n+1}(v) \\ F_n(v) \end{bmatrix} = \begin{bmatrix} F_{n+2}(v) \\ F_{n+1}(v) \end{bmatrix},$$

the standard argument implies that for $\lambda_{\pm}(v) = \frac{1 \pm \sqrt{4v-3}}{2}$, we have

$$F_n(v) = \frac{\lambda_+(v)^{n+1} - \lambda_-(v)^{n+1}}{\lambda_+(v) - \lambda_-(v)}.$$

One can also check directly that the right side satisfies the recurrence, using $\lambda_+\lambda_- = 1 - v$ and $\lambda_+ + \lambda_- = 1$.

Lemma 3.1. *For each $n \geq 2$, the solutions to $F_n(v) = 0$ are*

$$\xi_{n,k} = 1 - \frac{1}{2 + 2\cos(\frac{2\pi k}{n+1})}, \quad k = 1, \ldots, \left[\frac{n}{2}\right].$$

Proof. If $F_n(v) = 0$ then

$$\lambda_+(v)^{n+1} = \lambda_-(v)^{n+1}$$

so $\lambda_+(v) = \zeta\lambda_-(v)$ with $\zeta^{n+1} = 1$ and $\zeta \neq \pm 1$. Thus $\zeta\lambda_-(v) + \lambda_-(v) = 1$, so $\lambda_-(v) = \frac{1}{1+\zeta}$ and

$$1 - v = \lambda_+\lambda_- = \frac{\zeta}{(1+\zeta)^2}.$$

Hence

$$v = \frac{1 + \zeta + \zeta^2}{1 + 2\zeta + \zeta^2} = \frac{1 + \zeta + \zeta^{-1}}{2 + \zeta + \zeta^{-1}}.$$

Now substituting $\zeta = \left(e^{\frac{2\pi i}{n+1}}\right)^k$ the lemma follows. $\square$

Note that $\xi_{2n,n}$ is the leftmost zero of F_{2n} and $\xi_{2n+1,n}$ is the leftmost zero of F_{2n+1}. The following properties of $\xi_{n,k}$ are easily checked.

Lemma 3.2. *We have*
$\xi_{n,k} < \xi_{n,l}$ *if* $k > l$,
$\xi_{2n,n} < \xi_{2m,m}$ *and* $\xi_{2n+1,n} < \xi_{2m+1,m}$ *if* $n > m$, *and*
$\xi_{2n,n} < \xi_{2n+1,n} < \xi_{2n+2,n}$.

Set $v_n = \xi_{2n,n}$ for $n \geq 1$. This definition of v_n agrees with that given in Section 1. Recall that $0 = v_1 > v_2 > \ldots$.

Lemma 3.3. *If $v_{k+1} < v < v_k$ then*

$$(-1)^{\left[\frac{n}{2}\right]} F_n(v) > 0, \quad 0 \leq n \leq 2k+1, \tag{3.1}$$

$$(-1)^k F_{2k+2}(v) > 0. \tag{3.2}$$

Moreover, (3.1) *holds for all $v < v_k$.*

Proof. Let $v < v_k$. Then Lemma 3.2 implies that v is to the left of all the zeros of F_n for $n \le 2k+1$. Since the recurrence implies that the polynomial F_n has degree $\left[\frac{n}{2}\right]$ with positive leading coefficient, (3.1) follows. When also $v > v_{k+1}$, (3.2) holds because by Lemma 3.2, v is to the right of exactly one zero of F_{2k+2}. □

We are now ready to prove the main result of this section.

Theorem 3.4. *Let $v_{k+1} \le v < v_k$. For $1 \le i \le 2k+1$, define*

$$p_i := \frac{(-1)^{i+1} F_{i-1}(v) F_{2k+1-i}(v)}{F_{2k+1}(v)}.$$

If $v_{k+1} < v < v_k$, then all $p_i > 0$ and the unique optimal strategy is to play i with probability p_i for $1 \le i \le 2k+1$. If $v = v_{k+1}$, then this strategy is still optimal, but it is not unique, since it is also optimal to play $i+1$ with probability p_i for $1 \le i \le 2k+1$.

Proof. By Lemma 3.3, the p_i are all well-defined positive numbers. If $v_{k+1} < v < v_k$, then appealing again to Lemma 3.3, we see that the three conditions of Proposition 2.2 are satisfied. Proposition 2.2 thus shows that playing i with probability p_i for $1 \le i \le 2k+1$ is the unique optimal strategy. Now suppose that $v = v_{k+1}$. Then this strategy is still optimal, but it is not unique, since by the argument above with $k+1$ in place of k, it is also optimal to play j with probability

$$q_j := \frac{(-1)^{j+1} F_{j-1}(v) F_{2k+3-j}(v)}{F_{2k+3}(v)}$$

for $1 \le j \le 2k+3$. Observe that $q_1(v) = q_{2k+3}(v) = 0$, since $F_{2k+2}(v) = 0$. It remains to show that $p_i = q_{i+1}$. This can be proved by induction on i, using the recurrence for F_n. □

4. Strategies for some constricted multivariate games

For $k \ge 1$, let V_k be the set of infinite vectors $(x_1, x_2, \dots)$ with $x_i < 0$ for all $i \ge 1$ that satisfy the three conditions of Proposition 2.2. When $(x_1, x_2, \dots) \in V_k$, Proposition 2.2 describes the unique optimal strategy for the corresponding game. The uniqueness assertion implies that $V_i \cap V_j = \emptyset$ for $i \ne j$.

Note that by Lemma 3.3 and Theorem 3.4, $(v, v, v, \dots) \in V_k$ if and only if $v_{k+1} < v < v_k$. We use this fact to give a class of multivariate games with unique optimal strategy, in Theorem 4.2.

Proposition 4.1. *Suppose that $v_{k+1} < v < v_k$. Then there exists $\varepsilon > 0$ such that if*

$$|x_i - v| < \varepsilon \text{ for } i = 1, \dots, 2k+2$$

then $(x_1, x_2, \dots) \in V_k$.

Proof. We have $(v, v, v, \dots) \in V_k$ and V_k is open in $\mathbb{R}^{\infty}_{<0}$. □

We next determine V_1. The conditions defining this set are $x_i < 0$ for all $i \ge 1$ and

$$\frac{x_2}{x_1 + x_2 - 1} > 0, \; \frac{-1}{x_1 + x_2 - 1} > 0, \; \frac{x_1}{x_1 + x_2 - 1} > 0, \; \frac{x_1 x_3 + x_2 - 1}{x_1 + x_2 - 1} > 0.$$

All of the conditions but the last are automatic if the x_i are all negative. Thus

$$V_1 = \{(x_1, x_2, \dots) \in \mathbb{R}^{\infty}_{<0} \mid x_2 < \min\{0, 1 - x_1 x_3\}\}.$$

In particular, if C denotes the interior of a unit cube with vertices $-(a,b,c)$, $a,b,c \in \{0,1\}$ then $C \times \mathbb{R}^{\infty}_{<0}$ is contained in V_1. The following theorem extends this, by giving for every $m = 1,2,\dots,$ an open unit hypercube C_{2m+1} in $\mathbb{R}^{2m+1}_{<0}$ such that $C_{2m+1} \times \mathbb{R}^{\infty}_{<0}$ is contained in V_m. We will prove:

Theorem 4.2. *For $m = 1,2,\dots,$ let*

$$u_m = \frac{2\cos(\frac{\pi}{m+1})}{\cos(\frac{\pi}{m+1}) - 1}.$$

If $(x_1, x_2, \dots) \in \mathbb{R}^{\infty}_{<0}$ satisfies

$$u_m > x_i > u_m - 1, \quad i = 1,\dots,2m+1$$

then $(x_1, x_2, \dots) \in V_m$. In particular, Proposition 2.2 describes the unique optimal strategy for the corresponding game.

The proof will occupy the rest of the section. We start with the following lemmas.

Lemma 4.3. *Let w be an indeterminate. Then fixing $w^{\frac{1}{2}}$, we have*

(4.1)

$$F_{2n}(w+1, w, w+1, \dots, w, w+1) = \frac{1}{2}(w^{\frac{1}{2}})^{n-1}((w^{\frac{1}{2}}+1)^{n+1} + (w^{\frac{1}{2}}-1)^{n+1}), \quad n \geq 0,$$

$$F_{2n}(w, w+1, w, \dots, w+1, w) = wF_{2n-2}(w+1, w, w+1, \dots, w, w+1), \quad n \geq 1. \tag{4.2}$$

Proof. Let h_n denote the right side of (4.1). Direct calculation shows that

$$h_{n+2} = 2wh_{n+1} - w(w-1)h_n.$$

The left side of (4.1) satisfies the same recurrence, by Lemma 5.4. Since both sides equal 1 for $n = 0$ and $1 + w$ for $n = 1$, we obtain (4.1). Each side of (4.2) also satisfies the recurrence above. Since both sides equal w for $n = 0$ and $w(w + 1)$ for $n = 1$, we obtain (4.2). □

Lemma 4.4. *For fixed $m > 0$, set $u = u_m$ and $w = u_m - 1$, in the notation of Theorem 4.2. Then for $0 \leq n < m$*

$$(-1)^n F_{2n}(u, w, u, w, \dots, u) > 0, \tag{4.3}$$

$$F_{2m}(u, w, u, w, \dots, u) = 0, \tag{4.4}$$

$$F_{2m+2}(w, u, w, u, \dots, w) = 0. \tag{4.5}$$

Proof. We note that if

$$t = \frac{\cos(\frac{\pi}{m+1}) + 1}{\sin(\frac{\pi}{m+1})}$$

then

$$t^2 = \frac{\cos(\frac{\pi}{m+1}) + 1}{1 - \cos(\frac{\pi}{m+1})} = -w.$$

By Lemma 4.3 with $w^{\frac{1}{2}} = it$,

$$F_{2n}(u, w, u, w, \dots, u) = \frac{1}{2}(it)^{n-1}((it+1)^{n+1} + (it-1)^{n+1}).$$

After some simplification, the right side reduces to $\frac{1}{2}(-1)^n \frac{(\cos\frac{\pi}{m+1}+1)^{n-1}}{(\sin\frac{\pi}{m+1})^{2n}} H$, where

$$H = (1+\zeta^2)^{n+1} + (1+\zeta^{-2})^{n+1} = (\zeta+\zeta^{-1})^{n+1}(\zeta^{n+1}+\zeta^{-n-1}), \quad \zeta = e^{\frac{\pi i}{2m+2}}.$$

Thus, to prove (4.3), we must show that $H > 0$. If $n < m$, these factors in H involve positive cosines, while if $n = m$, the rightmost factor in H vanishes. This proves (4.3) and (4.4). Finally, (4.5) follows from (4.2). □

We are now ready to prove Theorem 4.2. Assume that $u_m - 1 < x_i < u_m$ for $1 \le i \le 2m+1$. To satisfy the three conditions of Proposition 2.2, it suffices to prove that

$$(-1)^{\left[\frac{h}{2}\right]} F_h(x_1, \ldots, x_{h-1}) > 0, \quad 0 \le h \le 2m+1$$

and

$$(-1)^m F_{2m+2}(x_1, \ldots, x_{2m+1}) > 0.$$

It is convenient to work with $G_n(x_1, \ldots, x_{n-1}) := (-1)^{\left[\frac{n}{2}\right]} F_n(x_1, \ldots, x_{n-1})$. Lemma 5.7 implies that for $1 \le j < 2n$ we have

$$(-1)^j \frac{\partial}{\partial x_j} G_{2n}(x_1, \ldots, x_{2n-1}) = G_{j-1}(x_1, \ldots, x_{j-2}) G_{2n-j-1}(x_{j+2}, \ldots, x_{2n-1}).$$

We first use this formula to prove by induction that $G_h(x_1, \ldots, x_{h-1}) > 0$ for $0 \le h \le 2m+1$. Clearly this holds for $h = 0$ and $h = 1$. Assume that it holds for all $h < 2n$ for some n with $1 \le n \le m$. We will prove that it holds for $h = 2n$ and $h = 2n+1$. By the induction hypothesis and the derivative formula above, $G_{2n}(x_1, \ldots, x_{2n-1})$ is strictly decreasing in x_1, strictly increasing in x_2, strictly decreasing in x_3, etc. Hence

$$G_{2n}(x_1, \ldots, x_{2n-1}) > G_{2n}(u_m, u_m - 1, \ldots, u_m, u_m - 1, u_m) \ge 0$$

by (4.3) and (4.4). This proves the result for $h = 2n$. By the induction hypothesis,

$$G_{2n+1}(x_1, \ldots, x_{2n}) = G_{2n}(x_1, \ldots, x_{2n-1}) + (1 - x_{2n}) G_{2n-1}(x_1, \ldots, x_{2n-2}) > 0,$$

so the result holds for $h = 2n+1$ as well.

It remains to prove that $G_{2m+2}(x_1, \ldots, x_{2m+1}) < 0$. Applying Lemma 5.7 again we find that G_{2m+2} has the same monotonicity properties (decreasing in the odd variables, increasing in the even ones), hence

$$G_{2m+2}(x_1, \ldots, x_{2m+1}) < G_{2n}(u_m - 1, u_m, \ldots, u_m - 1, u_m, u_m - 1) = 0,$$

by (4.5). This completes the proof of Theorem 4.2.

5. Appendix: Pfaffians associated with payoff matrices

In this section we will analyze the following skew-symmetric $m \times m$ matrices over a field F of characteristic 0:

$$A_m = A_m(x_1, \ldots, x_{m-1}) = \begin{bmatrix} 0 & x_1 & 1 & 1 & \ldots & 1 & 1 \\ -x_1 & 0 & x_2 & 1 & \ldots & 1 & 1 \\ -1 & -x_2 & 0 & x_3 & \ldots & 1 & 1 \\ -1 & -1 & -x_3 & 0 & \ldots & 1 & 1 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ -1 & -1 & -1 & -1 & \ldots & 0 & x_{m-1} \\ -1 & -1 & -1 & -1 & \ldots & -x_{m-1} & 0 \end{bmatrix}.$$

Here the superdiagonal has indeterminate entries $x_1, x_2, \ldots, x_{m-1}$ and all of the entries above the superdiagonal are 1's. In Theorem 5.8 below, we determine $\ker A_{2k+1}$ when no x_i equals 1, and we express the Pfaffians of the diagonal minors of A_{2k+1}

in terms of the polynomials $F_n(x_1, \ldots, x_{n-1})$ defined in Section 1. We first need to recall some material about Grassmann algebras.

Let V be an m-dimensional vector space over F with choice of non-zero element Ω_m in the one-dimensional space $\bigwedge^m V$. With this choice there is natural isomorphism T of $\bigwedge^{m-1} V$ to the dual V^* given by the formula

$$x \wedge \eta = T(\eta)(x)\Omega_m$$

for $\eta \in \bigwedge^{m-1} V$ and $x \in V$. Let $e_1, \ldots, e_m$ be a basis of V so that $\Omega_m = e_1 \wedge \cdots \wedge e_m$. A basis of $\bigwedge^{m-1} V$ is given by the elements $e_1 \wedge \cdots \wedge \widehat{e_j} \cdots \wedge e_m$, where the circumflex indicates deletion. Thus if $x = \sum_{j=1}^m x_j e_j$ and if $\eta = \sum_{j=1}^m \eta_j (e_1 \wedge \cdots \wedge \widehat{e_j} \cdots \wedge e_m)$ then $T(\eta)(x) = \sum_{j=1}^m (-1)^{j-1} \eta_j x_j$.

If A is a skew-symmetric matrix of size $m \times m$ with entries a_{ij} then we define

$$\omega_A = \sum_{i<j} a_{ij} e_i \wedge e_j.$$

We note if g is an $m \times m$ matrix with transpose g^T then

$$\omega_{gAg^T} = \left(\bigwedge^2 g\right) \omega_A$$

where

$$\left(\bigwedge^k g\right)(v_1 \wedge \cdots \wedge v_k) = gv_1 \wedge \cdots \wedge gv_k.$$

If $m = 2n$ with n an integer, then the Pfaffian of A, $Pf(A)$, is given by the formula

$$\frac{\omega_A^n}{n!} = Pf(A)\Omega_{2n};$$

here the n-th power is in the Grassmann algebra. If A is a $2n+1 \times 2n+1$ skew-symmetric matrix then $\frac{\omega_A^n}{n!}$ is in $\bigwedge^{2n} F^{2n+1}$. Thus, as above, we have an element $T(\frac{\omega_A^n}{n!}) \in \left(F^{2n+1}\right)^*$. Using the standard form $(x, y) = \sum x_i y_i$, we can identify $\left(F^{2n+1}\right)^*$ with F^{2n+1}.

If A is a matrix then we denote by $A_{r,s}$ the matrix gotten by deleting the r-th row and the s-th column. Note that when A is skew-symmetric, so is A_{rr}. The following lemma is standard but not easily referenced.

Lemma 5.1. *Let A be a $2n+1 \times 2n+1$ skew symmetric matrix. Then, using the standard form to view $T(\frac{\omega_A^n}{n!})$ as an element in F^{2n+1}, we have*

$$\text{(5.1)} \qquad A \text{ is of rank } 2n \text{ if and only if } \frac{\omega_A^n}{n!} \neq 0. \text{ Furthermore, } AT(\frac{\omega_A^n}{n!}) = 0.$$

Also, as an element of F^{2n+1},

$$\text{(5.2)} \qquad T(\frac{\omega_A^n}{n!}) = \sum_{i=1}^{2n+1} (-1)^{i+1} Pf(A_{ii}) e_i.$$

Proof. We note that there exists $g \in GL(2n+1, F)$ such that

$$\omega_{gAg^T} = \bigwedge^2 g \ \omega_A = \sum_{i=1}^l e_{2i-1} \wedge e_{2i}$$

with $2l$ equal to the rank of A. We therefore see that $\frac{\omega_A^n}{n!} \neq 0$ if and only if $l = n$. To see that

$$AT(\frac{\omega_A^n}{n!}) = 0$$

it is enough to show that

$$(Ax) \wedge \frac{\omega_A^n}{n!} = 0$$

for all $x \in F^{2n+1}$. For g as above, we must show that

$$0 = \left(\bigwedge^{2n+1} g\right)(Ax) \wedge \frac{\omega_A^n}{n!} = (gAx) \wedge \frac{\omega_{gAg^T}^n}{n!}.$$

That is, we must show that for all x,

$$((gAg^T)(g^T)^{-1}x) \wedge \frac{\omega_{gAg^T}^n}{n!} = 0.$$

This follows because the image of gAg^T is contained in the span of $\{e_1, e_2, \ldots, e_{2l}\}$ and $\frac{\omega_{gAg^T}^n}{n!}$ is either zero or a nonzero scalar multiple of Ω_{2n}. This proves (5.1).

For each j we write

$$\omega_A = \omega_j + \sum_{i=1}^{2n+1} a_{ij} e_i \wedge e_j = \omega_j + \beta_j.$$

Then we note that ω_j is $\omega_{A_{jj}}$ in the basis $e_1, \ldots, e_{j-1}, e_{j+1}, \ldots, e_{2n+1}$, and $\beta_j \wedge \beta_j = 0$. Thus

$$\frac{\omega_A^n}{n!} = \frac{\omega_j^n}{n!} + n\frac{\omega_j^{n-1}}{n!}\beta_j.$$

Since the last term is a multiple of e_j in the Grassmann algebra, we see that the coefficient of $e_1 \wedge \cdots \wedge \widehat{e_j} \cdots \wedge e_{2n+1}$ is $Pf(A_{jj})$. This proves (5.2). $\square$

Set

$$f_0 = 1, \quad f_n = f_n(x_1, \ldots, x_{2n-1}) = Pf(A_{2n}(x_1, \ldots, x_{2n-1})), \quad n \geq 1.$$

Lemma 5.2. *We have*

$$f_n = x_{2n-1} f_{n-1} + (x_{2n-2} - 1) f_{n-2} + (x_{2n-2} - 1)(x_{2n-4} - 1) f_{n-3} + \cdots +$$
$$(x_{2n-2} - 1) \cdots (x_4 - 1) f_1 + (x_{2n-2} - 1) \cdots (x_4 - 1)(x_2 - 1).$$

Proof. Before working with f_n, we investigate properties of the following expressions:

$$\mu_n = \sum_{i=1}^{2n-1} x_i e_i \wedge e_{i+1},$$

$$\gamma_n = \sum_{1 \leq i < j-1 < 2n} e_i \wedge e_j,$$

$$\nu_n = x_{2n-2} e_{2n-2} \wedge e_{2n-1} + x_{2n-1} e_{2n-1} \wedge e_{2n},$$

$$\xi_n = \sum_{i \leq 2n-3} e_i \wedge e_{2n-1} + \sum_{i \leq 2n-2} e_i \wedge e_{2n}$$

and

$$\delta_{j,2n} = \sum_{i \leq j} e_i \wedge e_{j+1} \wedge e_{j+2} \wedge \cdots \wedge e_{2n}.$$

We have

$$\omega_{A_{2n}} = \mu_n + \gamma_n$$

with

$$\mu_n = \mu_{n-1} + \nu_n$$

and

$$\gamma_n = \gamma_{n-1} + \xi_n.$$

We will write the Grassmann multiplication of elements in the (commutative) even part of the Grassmann algebra without the wedge. We note that

$$\nu_n^2 = 0$$

and

$$\xi_n^2 = 2\sum_{i\leq 2n-3}\sum_{j\leq 2n-2} e_i\wedge e_{2n-1}\wedge e_j\wedge e_{2n} =$$

$$-2\sum_{i\leq 2n-3}\sum_{j\leq 2n-2} e_i\wedge e_j\wedge e_{2n-1}\wedge e_{2n} =$$

$$-2\sum_{i\leq 2n-3}\sum_{j\leq 2n-3} e_i\wedge e_j\wedge e_{2n-1}\wedge e_{2n} \ -2\sum_{i\leq 2n-3} e_i\wedge e_{2n-2}\wedge e_{2n-1}\wedge e_{2n} =$$

$$-2\sum_{i\leq 2n-3} e_i\wedge e_{2n-2}\wedge e_{2n-1}\wedge e_{2n}$$

since the first sum in the penultimate expression is 0. We write this as

$$\xi_n^2 = -2\delta_{2n-3,2n}.$$

Also

$$\nu_n\xi_n = x_{2n-2}\delta_{2n-3,2n}.$$

Similarly, for $1 < j < n$, one calculates

$$(\nu_j+\xi_j)^2\delta_{2j-1,2n} = 0$$

and

$$(\nu_j+\xi_j)\delta_{2j-1,2n} = (x_{2j-2}-1)\delta_{2j-3,2n}.$$

We are now ready to derive the formula for f_n. We have

$$f_n\Omega_{2n} = \frac{(\mu_n+\gamma_n)^n}{n!} =$$

$$\frac{(\mu_{n-1}+\gamma_{n-1}+\nu_n+\xi_n)^n}{n!} =$$

$$\frac{(\mu_{n-1}+\gamma_{n-1})^n}{n!} + n\frac{(\mu_{n-1}+\gamma_{n-1})^{n-1}}{n!}(\nu_n+\xi_n) +$$

$$\binom{n}{2}\frac{(\mu_{n-1}+\gamma_{n-1})^{n-2}}{n!}(\nu_n+\xi_n)^2 = C_1+C_2+C_3.$$

Since C_1 is of degree $2n$ in $e_1,\ldots,e_{2n-2}$ it is 0. We have

$$C_2 = f_{n-1}\Omega_{2n-2}(\nu_n+\xi_n) = x_{2n-1}f_{n-1}\Omega_{2n}.$$

We now look at C_3. Since

$$(\nu_n+\xi_n)^2 = 2\nu_n\xi_n + \xi_n^2 = 2(x_{2n-2}-1)\delta_{2n-3,2n},$$

we have

$$C_3 = (x_{2n-2} - 1)\frac{(\mu_{n-1} + \gamma_{n-1})^{n-2}}{(n-2)!}\delta_{2n-3,2n}.$$

We now have our "bootstrap":

$$\frac{(\mu_{n-1} + \gamma_{n-1})^{n-2}}{(n-2)!}\delta_{2n-3,2n} = \frac{(\mu_{n-2} + \gamma_{n-2} + \nu_{n-1} + \xi_{n-1})^{n-2}}{(n-2)!}\delta_{2n-3,2n}$$

$$= \frac{(\mu_{n-2} + \gamma_{n-2})^{n-2}}{(n-2)!}\delta_{2n-3,2n} + (n-2)\frac{(\mu_{n-2} + \gamma_{n-2})^{n-3}}{(n-2)!}(\nu_{n-1} + \xi_{n-1})\delta_{2n-3,2n} =$$

$$f_{n-2}\Omega_{2n} + \frac{(\mu_{n-2} + \gamma_{n-2})^{n-3}}{(n-3)!}(x_{2n-4} - 1)\delta_{2n-5,2n}.$$

Now repeat the argument on the second term, and continue in this manner, to obtain Lemma 5.2. □

The next result simplifies the recurrence relation for f_n.

Proposition 5.3. *We have*

$$f_n = (x_{2n-1} + x_{2n-2} - 1)f_{n-1} - (x_{2n-2} - 1)(x_{2n-3} - 1)f_{n-2}$$

with $f_0 = 1$ *and* $f_1 = x_1$.

Proof. The initial conditions are clear. We write the formula in Lemma 5.2 as

$$f_n = x_{2n-1}f_{n-1} + (x_{2n-2} - 1)(f_{n-2} + (x_{2n-4} - 1)f_{n-3} + \cdots +$$

$$(x_{2n-4} - 1)\cdots(x_4 - 1)f_1 + (x_{2n-4} - 1)\cdots(x_4 - 1)(x_2 - 1)).$$

This expression is (applying Lemma 5.2 with $n-1$ replacing n)

$$f_n = x_{2n-1}f_{n-1} + (x_{2n-2} - 1)((1 - x_{2n-3})f_{n-2} + f_{n-1}) =$$

$$(x_{2n-1} + x_{2n-2} - 1)f_{n-1} - (x_{2n-2} - 1)(x_{2n-3} - 1)f_{n-2},$$

as asserted. □

Define the polynomials

$$F_{2n} = F_{2n}(x_1, \ldots, x_{2n-1}) = f_n(x_1, \ldots, x_{2n-1})$$

and

$$F_{2n+1} = F_{2n+1}(x_1, \ldots, x_{2n}) = f_{n+1}(x_1, \ldots, x_{2n}, 1).$$

In particular, $F_{2m}(x_1, \ldots, x_{2m-1})$ equals the Pfaffian of $A_{2m}(x_1, \ldots, x_{2m-1})$. The following result shows that these F_m are the same multivariate polynomials that were defined by the recurrence in Section 1.

Lemma 5.4. *The polynomials* $F_n(x_1, \ldots, x_{n-1})$ *as defined above are the solution to the recurrence relation*

$$F_{-1} = 0,\ F_0 = F_1 = 1$$

and

$$F_{n+2} = F_{n+1} + (x_{n+1} - 1)F_n, \quad n \geq 0.$$

Furthermore,

$$F_{n+2} = (x_{n+1} + x_n - 1)F_n - (x_n - 1)(x_{n-1} - 1)F_{n-2}, \quad n \geq 1.$$

Proof. We begin by proving the first recurrence in the even case $n = 2k$. Writing $f_n = f_n(x_1, \ldots, x_{2n-1})$, we have

$$F_{2k+1}(x_1, \ldots, x_{2k}) = f_{k+1}(x_1, \ldots, x_{2k}, 1) =$$
$$x_{2k} f_k - (x_{2k} - 1)(x_{2k-1} - 1) f_{k-1} =$$
$$(x_{2k+1} + x_{2k} - 1) f_k - (x_{2k} - 1)(x_{2k-1} - 1) f_{k-1} - (x_{2k+1} - 1) f_k =$$
$$f_{k+1} - (x_{2k+1} - 1) f_k = F_{2k+2} - (x_{2k+1} - 1) F_{2k}.$$

Thus

$$F_{2k+2} = F_{2k+1} + (x_{2k+1} - 1) F_{2k}.$$

We next look at the odd case $n = 2k + 1$. Then

$$F_{2k+3}(x_1, \ldots, x_{2k+2}) = f_{k+2}(x_1, \ldots, x_{2k+2}, 1) =$$
$$x_{2k+2} f_{k+1} - (x_{2k+2} - 1)(x_{2k+1} - 1) f_k =$$
$$f_{k+1} + (x_{2k+2} - 1)(f_{k+1} - (x_{2k+1} - 1) f_k).$$

Now $f_{k+1} - (x_{2k+1} - 1) f_k = F_{2k+1}(x_1, \ldots, x_{2k})$ by the first part of this argument. Hence

$$F_{2k+3} = F_{2k+2} + (x_{2k+2} - 1) F_{2k+1}.$$

Since the initial values are obvious, this completes the proof of the first recurrence.

To prove the second recurrence, note that

$$F_{n+2} = F_{n+1} + (x_{n+1} - 1) F_n = F_n + (x_n - 1) F_{n-1} + (x_{n+1} - 1) F_n =$$
$$(x_{n+1} + x_n - 1) F_n + (x_n - 1)(-F_n + F_{n-1}) =$$
$$(x_{n+1} + x_n - 1) F_n - (x_n - 1)(x_{n-1} - 1) F_{n-2}.$$

This completes the proof of the second recurrence. □

Lemma 5.5. *Let $n \geq 2$. For $1 \leq i \leq n - 1$,*

$$F_n(x_1, \ldots, x_{i-1}, 1, x_{i+1}, \ldots, x_{n-1}) = F_i(x_1, \ldots, x_{i-1}) F_{n-i}(x_{i+1}, \ldots, x_{n-1}).$$

Proof. We prove this by induction on n. If $n = 2$, this says that $1 = F_1 F_1$, which is true. We have

$$F_n(x_1, \ldots, x_{n-2}, 1) = F_{n-1} + (1 - 1) F_{n-2} = F_{n-1}.$$

Since $F_1 = 1$ this proves the formula for $i = n - 1$. Now

$$F_n(x_1, \ldots, x_{n-3}, 1, x_{n-1}) = F_{n-1}(x_1, \ldots, x_{n-3}, 1) + (x_{n-1} - 1) F_{n-2} =$$
$$F_{n-2} + (x_{n-1} - 1) F_{n-2} = F_{n-2} F_2(x_{n-1}).$$

This proves the formula for $i = n - 2$. In particular, we now know the formula is valid for $n = 3$. Suppose that $i < n - 2$ with $n > 3$. Then

$$F_n(x_1, \ldots, x_{i-1}, 1, x_{i+1}, \ldots, x_{n-1}) = F_{n-1}(x_1, \ldots, x_{i-1}, 1, x_{i+1}, \ldots, x_{n-2})$$
$$+ (x_{n-1} - 1) F_{n-2}(x_1, \ldots, x_{i-1}, 1, x_{i+1}, \ldots, x_{n-3}).$$

The induction hypothesis implies that this is equal to

$$F_i(x_1, \ldots, x_{i-1})(F_{n-1-i}(x_{i+1}, \ldots, x_{n-2}) + (x_{n-1} - 1) F_{n-2-i}(x_{i+1}, \ldots, x_{n-3}))$$
$$= F_i(x_1, \ldots, x_{i-1}) F_{n-i}(x_{i+1}, \ldots, x_{n-1}).$$

□

We now examine $F_{2k+1} = F_{2k+1}(x_1, \ldots, x_{2k})$ and $A_{2k+1} = A_{2k+1}(x_1, \ldots, x_{2k})$.

Lemma 5.6. *We have*

$$T\left(\frac{\omega^k_{A_{2k+1}}}{k!}\right) = \sum_{i=1}^{2k+1} (-1)^{i+1} F_{i-1}(x_1,\dots,x_{i-2})F_{2k+1-i}(x_{i+1},\dots,x_{2k})e_i.$$

Furthermore

$$F_{2k+1}(x_1,\dots,x_{2k}) = \sum_{i=1}^{2k+1} (-1)^{i+1} F_{i-1}(x_1,\dots,x_{i-2})F_{2k+1-i}(x_{i+1},\dots,x_{2k}).$$

Proof. For $i = 1,\dots,2k+1$, we consider $A_{2k+1}(x_1,\dots,x_{2k})_{ii}$ (with notation as in Lemma 5.1). One checks that if $i = 1$ then

$$A_{2k+1}(x_1,\dots,x_{2k})_{11} = A_{2k}(x_2,\dots,x_{2k})$$

and if $i = 2k+1$ then

$$A_{2k+1}(x_1,\dots,x_{2k})_{2k+1,2k+1} = A_{2k}(x_1,\dots,x_{2k-1}).$$

For $1 < i < 2k+1$ we have

$$A_{2k+1}(x_1,\dots,x_{2k})_{ii} = A_{2k}(x_1,\dots,x_{i-2},1,x_{i+1},\dots,x_{2k}).$$

The first part of the result now follows from Lemmas 5.1 and 5.5 in light of

$$Pf(A_{2k}(x_1,\dots,x_{2k-1})) = F_{2k}(x_1,\dots,x_{2k-1}).$$

We now turn to the formula for F_{2k+1}. This formula is easily checked for $k = 0,1,2$. Let $k \geq 3$. We will induct on k. By the second part of Lemma 5.4, we have

$$F_{2k+1}(x_1,\dots,x_{2k}) = (x_{2k} + x_{2k-1} - 1)F_{2k-1} - (x_{2k-1} - 1)(x_{2k-2} - 1)F_{2k-3}.$$

We now apply the induction hypothesis to F_{2k-1} and F_{2k-3} to see that F_{2k+1} equals

$$\sum_{i=1}^{2k+1} (-1)^{i+1} F_{i-1}\left((x_{2k} + x_{2k-1} - 1)H_{2k-1-i} - (x_{2k-1} - 1)(x_{2k-2} - 1)H_{2k-3-i}\right)$$

where we ignore all terms in which negative subscripts occur, and where $H_{2k-1-i} = F_{2k-1-i}(x_{i+1},\dots,x_{2k-2})$ and $H_{2k-3-i} = F_{2k-3-i}(x_{i+1},\dots,x_{2k-4})$. Applying the second part of Lemma 5.4 with $n = 2k-1-i$ for each $i \leq 2k-2$, we readily complete the induction. □

The following lemma will be used in the proof of Theorem 4.2.

Lemma 5.7. *For $1 \leq i \leq m$*

$$\frac{\partial}{\partial x_i} F_{m+1}(x_1,\dots,x_m) = F_{i-1}(x_1,\dots,x_{i-2})F_{m-i}(x_{i+2},\dots,x_m).$$

Proof. This result is proved by essentially the same argument as in the proof of Lemma 5.5. □

Theorem 5.8. *Suppose that $x_i \neq 1$ for all $i \geq 1$. Write $A = A_{2k+1}(x_1,\dots,x_{2k})$. Then $\ker A = \mathbb{R}\mathbf{t}$, where $\mathbf{t} = (t_1,\dots,t_{2k+1})$ with*

$$t_i = Pf(A_{ii}) = F_{i-1}(x_1,\dots,x_{i-2})F_{2k+1-i}(x_{i+1},\dots,x_{2k}), \quad 1 \leq i \leq 2k+1.$$

Proof. By Lemmas 5.1 and 5.6, the vector $\mathbf{t}$ described above lies in $\ker A$, and so by Lemma 5.1, it remains to show that $\mathbf{t}$ is nonzero. Assume that $\mathbf{t}$ is zero. Then $F_{2k}(x_1, \ldots, x_{2k-1}) = t_{2k+1}$ and $F_{2k-1}(x_1, \ldots, x_{2k-2}) = t_{2k}$ both vanish. But by the first recurrence for the sequence F_n in Lemma 5.4, the vanishing of two consecutive terms of the sequence implies the vanishing of all the terms, since $x_i \neq 1$ for all i. This contradicts the fact that $F_0 = 1$. □

We remark that Theorem 5.8 is false if one deletes the hypothesis that $x_i \neq 1$ for all $i \geq 1$. For example, $\ker A_5(0, 1, 0, 1)$ and $\ker A_5(0, 1, 1/2, 1/2)$ both have dimension 3.

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