FOURIER COEFFICIENTS OF HECKE EIGENFORMS

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Abstract

We provide systematic evaluations, in terms of binary quadratic representations of $4p$, for the p-th Fourier coefficients of each member f of an infinite class $\mathcal{C}$ of CM eigenforms. As an application, previously conjectured evaluations of three algebro-geometric character sums can now be formulated explicitly without reference to eigenforms. There are several non-CM newforms that appear to share some properties with the eigenforms in $\mathcal{C}$, and we pose some conjectures about their Fourier coefficients.

1 Introduction

Let $\mathbb{F}_p$ denote the field of p elements, where p is an odd prime. For $a \in \mathbb{F}_p^*$ and a multiplicative character C on $\mathbb{F}_p^*$, define the twisted Kloosterman sum

$$K(C^k, a) := \sum_{x \in \mathbb{F}_p^*} C^k(x) e^{2\pi i (x + a\overline{x})/p}, \tag{1.1}$$

where $\overline{x}$ is the inverse of $x \pmod{p}$. Let $g_k(a)$, $h_k(a)$ denote the zeros of the quadratic polynomial

$$X^2 + XK(C^k, a) + C^k(-a)p.$$

As in [4, (4.2)], consider the twisted sum of traces of the n-th symmetric power of twisted Kloosterman sheaves, defined by

$$T_n(C, k, \ell) := \sum_{a \in \mathbb{F}_p^*} \overline{C}^\ell(a) \left(g_k(a)^n + g_k(a)^{n-1} h_k(a) + \cdots + h_k(a)^n \right). \tag{1.2}$$

An estimate of this sum in the special case $k = \ell = 0$ may be found in [6, Theorem 4.6], while a generic estimate is displayed in [4, (4.8)]. In a number of special cases, precise determinations in terms of Fourier coefficients of Hecke eigenforms have been proved or conjectured [4]. For example, from [2, Theorem 2] and [4, p. 528], we have the following conjectured evaluation of the character sum $T_{15}(C, 1, 0)$ when $p \equiv 1 \pmod{3}$, $p > 15$, and C has order 3:

$$T_{15}(C, 1, 0) := p^5 r^5 - 4p^6 r^3 + 3p^7 r + \left(\frac{p}{105} \right) A_{15}(p), \tag{1.3}$$

where

$$4p = r^2 + 27t^2, \quad r \equiv 1 \pmod{3} \tag{1.4}$$

and

$$A_{15}(p) = \begin{cases} 2p^8 - p^7 |b_{15}(p)|^2, & \text{if } \left(\frac{1001}{p} \right) = 1 \\ 0, & \text{if } \left(\frac{1001}{p} \right) = -1, \end{cases} \tag{1.5}$$

for the p-th Fourier coefficient $b_{15}(p)$ of a weight 2 newform f_{15} for $\Gamma_0(39039)$ with quartic nebentypus of conductor 3003. There are similar conjectural evaluations for $T_7(C,2,1)$ and $T_{11}(C,1,1)$, in terms of weight 2 newforms f_7 and f_{11} for $\Gamma_0(175)$ and $\Gamma_0(5775)$, respectively.

In Section 3, we define a certain infinite class $\mathcal{C} \supset \{f_7, f_{11}, f_{15}\}$ of eigenforms attached to Hecke characters. (It would be interesting to find algebro-geometric objects associated with each of the CM newforms in $\mathcal{C}$.) Theorem 3.1 gives systematic evaluations of the p-th Fourier coefficients of each $f \in \mathcal{C}$, in terms of binary quadratic representations of $4p$. By virtue of this theorem, the conjectured evaluations of the aforementioned algebro-geometric character sums T_7, T_{11}, T_{15} can be formulated explicitly without any reference to eigenforms. For instance (see Example 3.5), when $p \equiv 1 \pmod 3$ and $p > 15$, formula (1.5) becomes

$$A_{15}(p) = \begin{cases} 2p^8 - p^7 E u^2, & \text{if } \left(\frac{1001}{p}\right) = 1 \\ 0, & \text{if } \left(\frac{1001}{p}\right) = -1, \end{cases} \tag{1.6}$$

where for p satisfying $\left(\frac{1001}{p}\right) = 1$,

$$E = \begin{cases} 1, & \text{if } \left(\frac{p}{7}\right) = 1, \quad \left(\frac{p}{11}\right) = 1 \\ 7, & \text{if } \left(\frac{p}{7}\right) = 1, \quad \left(\frac{p}{11}\right) = -1 \\ 33, & \text{if } \left(\frac{p}{7}\right) = -1, \quad \left(\frac{p}{11}\right) = 1 \\ 231, & \text{if } \left(\frac{p}{7}\right) = -1, \quad \left(\frac{p}{11}\right) = -1 \end{cases} \tag{1.7}$$

and the integer u^2 is uniquely defined by

$$4p = Eu^2 + Fv^2, \quad EF = 3003. \tag{1.8}$$

Let $\mathcal{D}$ denote the set of 62 integers 7, 11, 19, 43, 67, 163; 20, 24, 40, 52, 15, 88, 35, 148, 51, 232, 91, 115, 123, 187, 235, 267, 403, 427; 84, 120, 132, 168, 228, 280, 312, 340, 372, 408, 520, 532, 708, 760, 195, 1012, 435, 483, 555, 595, 627, 715, 795, 1435; 420, 660, 840, 1092, 1320, 1380, 1428, 1540, 1848, 1155, 1995, 3003, 3315; 5460. The imaginary quadratic field

$$K = \mathbb{Q}(\sqrt{-D}), \quad D \in \mathcal{D} \tag{1.9}$$

has fundamental discriminant $-D$, and the set of units in its ring of integers $\mathcal{O}_K$ is $\{\pm 1\}$. For each $D \in \mathcal{D}$, the class group of K has the same cardinality as the genus class group of K. Moreover, under the Generalized Riemann

Hypothesis, there are no other imaginary quadratic fields with this property, except for $\mathbb{Q}(\sqrt{-1})$, $\mathbb{Q}(\sqrt{-2})$, $\mathbb{Q}(\sqrt{-3})$; see [8].

This paper proceeds as follows. In Section 2, we introduce parameters E and u occurring in binary quadratic representations of $4p$, for primes p that split in $\mathcal{O}_K$. In Section 3, we define an infinite class $\mathcal{C}$ of eigenforms attached to Hecke characters on K. Theorem 3.1 provides an algorithm for systematically computing the p-th Fourier coefficient $b(p)$ of each eigenform $f \in \mathcal{C}$ in terms of the parameters E and u. A consequence of Theorem 3.1 (see (3.14)) is that $b(p)$ has the form

$$b(p) = m(p)\sqrt{\chi(p)w(p)}, \tag{1.10}$$

where χ is a certain Dirichlet character related to f, and $m(p)$, $w(p)$ are nonnegative integers such that $w(p)$ depends only on p's signature (2.14). Conjecture 2.1 in [3] suggests that a phenomenon like (1.10) holds for a non-CM weight 3 newform for $\Gamma_0(525)$ with quartic nebentypus of conductor 105. Section 4 offers conjectures of this kind for several more non-CM newforms; see Table 2. For the corresponding integers $m(p)$, Conjectures 4.1 and 4.2 give congruences that depend only on p's signature. All of these conjectures have been verified for $p < 5000$, using a `Sage` program like the one described in [3].

2 Representation of primes p by binary quadratic forms

Throughout this and the next section, p and ℓ denote primes which are split and ramified, respectively, in $K = \mathbb{Q}(\sqrt{-D})$, where $D \in \mathcal{D}$. Write the prime ideal factorizations of (p) and (ℓ) in $\mathcal{O}_K$ as

$$(p) = \mathfrak{P}_p\overline{\mathfrak{P}}_p, \quad (l) = \mathfrak{P}_\ell^2. \tag{2.1}$$

Fix a distinguished odd prime factor r of D. Let

$$L = \{\ell_1, \ell_2, \dots, \ell_k\}, \quad \ell_1 < \cdots < \ell_k \tag{2.2}$$

denote the set of prime factors of D/r.

Since $D \in \mathcal{D}$, it follows from genus theory [7, p. 509] that for any prime p (that splits in K),

$$\mathfrak{P}_p \prod_{\ell \in L} \mathfrak{P}_\ell^{a(\ell)} = (\alpha) \tag{2.3}$$

for some $\alpha \in \mathcal{O}_K$ and some $a(\ell) \in \{0, 1\}$. Each of the 2^k ideal classes of K can be written in the form $[\mathfrak{P}_p]$ for some (split) p, and (2.3) yields a bijection between these 2^k ideal classes $[\mathfrak{P}_p]$ and the 2^k tuples $(a(\ell_1), \dots, a(\ell_k))$. Taking norms in (2.3), we have

$$pE = \alpha\overline{\alpha}, \quad E = \prod_{\ell \in L} \ell^{a(\ell)}. \tag{2.4}$$

Since E divides D/r, we can write

$$D = EF, \quad r \mid F. \tag{2.5}$$

As α is an integer of $K = \mathbb{Q}(\sqrt{-D})$, there are rational integers x, v for which

$$\alpha = (x + iv\sqrt{D})/2. \tag{2.6}$$

Then by (2.4),

$$4pE = x^2 + Dv^2. \tag{2.7}$$

By (2.5) and (2.7), E divides x, so we can write $x = Eu$ for some integer u. Consequently, (2.6) and (2.7) become

$$\alpha = (Eu + iv\sqrt{D})/2 \tag{2.8}$$

and

$$4p = Eu^2 + Fv^2, \quad D = EF, \tag{2.9}$$

for some integers u, v. Since $D > 4$, the integers u, v in (2.9) are unique up to sign for each (split) p [1, Lemma 3.01]. Observe that as p varies, E can be any one of the 2^k positive divisors of D for which $4 \nmid E$ and $r \nmid E$.

Write

$$-D = d_0 d_1 \cdots d_k, \tag{2.10}$$

where $d_0 = (-1)^{(r-1)/2} r$ and d_i is the prime discriminant corresponding to the prime ℓ_i, $i = 1, 2, \dots, k$. Define the functions ψ_i, $i = 0, 1, \dots, k$, on the set of primes q in terms of Kronecker symbols, as follows:

$$\psi_i(q) = \begin{cases} \left(\frac{d_i}{q}\right), & \text{if } q \nmid d_i \\ \left(\frac{-D/d_i}{q}\right), & \text{if } q \mid d_i. \end{cases} \tag{2.11}$$

Extend the definition by multiplicativity to define the functions ψ_i on all positive integers. It is easily seen that for any prime q that splits or ramifies in K,

$$\prod_{i=0}^{k} \psi_i(q) = 1. \tag{2.12}$$

For a given split prime p, we now describe a quick (well-known) method for determining the corresponding parameter E in (2.9). By (2.7), $\psi_i(pE) = 1$ for $i = 1, 2, \ldots, k$ (see for example [5, (2.12)]). Thus

$$\psi_i(E) = \psi_i(p) = \left(\frac{d_i}{p}\right), \quad i = 1, 2, \ldots, k. \tag{2.13}$$

Among the 2^k possible choices of E, only one will satisfy (2.13), in view of (2.12) and [5, (2.12)]. In other words E (and thus also F, $|u|$, and $|v|$) is completely determined by p's signature

$$\left(\left(\frac{d_1}{p}\right), \ldots, \left(\frac{d_k}{p}\right)\right). \tag{2.14}$$

Example 2.1. Let $D = 5460$, $r = 13$, $p = 1000003$. Then the signature (2.14) is

$$\left(\left(\frac{-4}{p}\right), \left(\frac{-3}{p}\right), \left(\frac{5}{p}\right), \left(\frac{-7}{p}\right)\right) = (-1, 1, -1, 1). \tag{2.15}$$

Examining the 16 possible values of E which divide 5460 with $4 \nmid E$, $13 \nmid E$, we find that (2.13) holds for $E = 42$. As a check,

$$4p = 4000012 = 42 \cdot 173^2 + 130 \cdot 51^2.$$

3 Eigenforms attached to Hecke characters

Choose a positive integer h and a Dirichlet character $\chi \pmod r$ for which $\chi(-1) = (-1)^h$. Note that χ has conductor r_1, where

$$r_1 = 1 \text{ or } r_1 = r \text{ according as } \chi \text{ is trivial or nontrivial.}$$

For $\alpha \in K$ prime to r, there is a unique nonzero rational integer $a \pmod r$ such that $\alpha \equiv a \pmod{\mathfrak{P}_r}$, so we can extend χ to a character on the elements of K prime to r by defining $\chi(\alpha) = \chi(a)$. We now create a complex-valued multiplicative function ϕ on the group of fractional ideals of K prime to r, by defining

$$\phi((\alpha)) = \alpha^h \chi(\alpha), \text{ for } \alpha \in K \text{ prime to } r, \tag{3.1}$$

$$\phi(\mathfrak{P}_\ell) = \left(\ell^h \chi(\ell)\right)^{1/2}, \text{ for } \ell \in L, \tag{3.2}$$

where we take a fixed branch of the square root, say the principal square root. By (2.3), this completely defines ϕ. Note that the square of (3.2) is consistent with (3.1), and ϕ is well-defined in (3.1) because $\chi(-1) = (-1)^h$. The function ϕ is a weight $h+1$ Hecke character on K. It gives rise to a weight $h+1$ eigenform f for $\Gamma_0(r_1 D)$ with nebentypus $\chi(\cdot)\left(\frac{-D}{\cdot}\right)$, namely

$$f(z) = \sum_{\mathfrak{A}} \phi(\mathfrak{A}) q^{N(\mathfrak{A})}, \quad q = e^{2\pi i z}, \tag{3.3}$$

where $\mathfrak{A}$ runs through all ideals of $\mathcal{O}_K$ whose norm $N(\mathfrak{A})$ is prime to r; see [9, p. 9]. These functions $f(z)$, for all possible choices of D, r, h, and χ, comprise an infinite class $\mathcal{C}$ of eigenforms attached to Hecke characters on K. For $f \in \mathcal{C}$, we wish to evaluate the Fourier coefficients $b(n)$ in

$$f(z) = \sum_{n=1}^{\infty} b(n) q^n. \tag{3.4}$$

By [6, (6.83)], it suffices to restrict our attention to prime n. Clearly

$$b(r) = 0, \quad b(n) = 0 \text{ for every prime } n \text{ inert in } K. \tag{3.5}$$

For ramified $n = \ell \in L$, it follows from (3.2) that

$$b(\ell) = \left(\ell^h \chi(\ell)\right)^{1/2}. \tag{3.6}$$

It remains to evaluate $b(p)$ for split p. Applying ϕ to (2.3), we have

$$\phi(\mathfrak{P}_p)\left(E^h \chi(E)\right)^{1/2} = \alpha^h \chi(\alpha), \tag{3.7}$$

where the branch of the square root is specified by

$$\left(E^h\chi(E)\right)^{1/2} = \prod_{\ell|E}\left(\ell^h\chi(\ell)\right)^{1/2}. \tag{3.8}$$

Since $\alpha \equiv \overline{\alpha} \pmod{\mathfrak{P}_r}$ by (2.8), we have

$$\chi(\alpha) = \chi(\overline{\alpha}) = \chi(uE/2). \tag{3.9}$$

Thus, applying ϕ to the complex conjugate of (2.3), we obtain

$$\phi(\overline{\mathfrak{P}_p})\left(E^h\chi(E)\right)^{1/2} = \overline{\alpha}^h\chi(\alpha). \tag{3.10}$$

Consequently,

$$b(p) = \phi(\mathfrak{P}_p) + \phi(\overline{\mathfrak{P}_p}) = (\alpha^h + \overline{\alpha}^h)\chi(\alpha)\left(E^h\chi(E)\right)^{-1/2}. \tag{3.11}$$

Therefore, by (2.8),

$$b(p) = 2\left(E^h\chi(E)\right)^{1/2}(u/2)^h\chi(u/2)\sum_{m\geq 0}\binom{h}{2m}\left(\frac{-v^2F}{Eu^2}\right)^m. \tag{3.12}$$

For brevity, write

$$G = Eu^2/4. \tag{3.13}$$

Since $\chi(G) = \chi(p)$ by (2.9), we can rewrite (3.12) as

$$b(p) = 2\sqrt{\chi(p)}\sum_{m\geq 0}\binom{h}{2m}(G-p)^m G^{h/2-m}, \tag{3.14}$$

where the branch of $\sqrt{\chi(p)} = \sqrt{\chi(G)}$ is determined by (3.8). We have thus proved the following theorem.

Theorem 3.1. *For $D \in \mathcal{D}$, let $K = \mathbb{Q}(\sqrt{-D})$. Choose any positive integer h, an odd prime factor r of D, and a Dirichlet character $\chi \pmod{r}$ of conductor r_1 with $\chi(-1) = (-1)^h$. As in (3.3), define a weight $h+1$ eigenform f for $\Gamma_0(r_1 D)$ with nebentypus $\chi(\cdot)\left(\frac{-D}{\cdot}\right)$ attached to a weight $h+1$ Hecke character on K. Let $\mathcal{C}$ denote the set of all such eigenforms f. Then the Fourier coefficients $b(n)$ of any $f \in \mathcal{C}$ are completely determined by (3.5), (3.6), and (3.14).*

Remark 3.2. Although 4 appears in the denominator of G in (3.13), it is nevertheless the case that the members of (3.14) are algebraic integers, by virtue of (3.11). It thus follows from (3.14) that when f has even weight (i.e., h is odd), $b(p)$ is an integer multiple of $\sqrt{\chi(p)E}$, whereas when f has odd weight, $b(p)$ is an integer multiple of $\sqrt{\chi(p)}$. Cf. (1.10).

Remark 3.3. The special cases of (3.14) for weights 2,3,4,5 are

$$b(p) = u\sqrt{\chi(p)E}, \text{ if } h = 1, \tag{3.15}$$

$$b(p) = \sqrt{\chi(p)}(Eu^2 - 2p), \text{ if } h = 2, \tag{3.16}$$

$$b(p) = u\sqrt{\chi(p)E}(Eu^2 - 3p), \text{ if } h = 3, \tag{3.17}$$

$$b(p) = \sqrt{\chi(p)}(E^2u^4 - 4pEu^2 + 2p^2), \text{ if } h = 4. \tag{3.18}$$

Remark 3.4. Theorem 3.1 shows that the coefficient field of any $f \in \mathcal{C}$ is

$$\begin{cases} \mathbb{Q}\left((\ell_1\chi(\ell_1))^{1/2}, \ldots, (\ell_k\chi(\ell_k))^{1/2}\right), & \text{if } h \text{ is odd} \\ \mathbb{Q}\left(\chi(\ell_1)^{1/2}, \ldots, \chi(\ell_k)^{1/2}\right), & \text{if } h \text{ is even.} \end{cases} \tag{3.19}$$

In particular, when f has odd weight, the coefficient field is cyclotomic.

Example 3.5. Choose $D = 3003$, $r = 13$, and $h = 1$. Let p denote any prime for which $\left(\frac{-3003}{p}\right) = 1$. Specify a quartic character $\chi \pmod{13}$ by setting $\chi(2) = i$. We have

$$4p = Eu^2 + Fv^2, \quad 13 \nmid E \tag{3.20}$$

for 8 possible factors E of $D = 3003$ depending on the choice of p. By the remarks above (2.14), these 8 possible factors correspond to the 8 possible signatures

$$s(p) := \left(\left(\frac{-3}{p}\right), \left(\frac{-7}{p}\right), \left(\frac{-11}{p}\right)\right). \tag{3.21}$$

Class $\mathcal{C}$ contains a weight 2 newform f for $\Gamma_0(39039)$ with a quartic nebentypus $\chi(\cdot)\left(\frac{-3003}{\cdot}\right)$ of conductor 3003. By (3.19), f has coefficient field

$$\mathbb{Q}(\sqrt{3}, \zeta\sqrt{7}, \zeta\sqrt{11}), \quad \zeta = e^{2\pi i/8} \tag{3.22}$$

of degree 16 over $\mathbb{Q}$. By (3.12) with $h = 1$, the p-th Fourier coefficients $b(p)$ of f are as given in Table 1. For primes n starting with $n = 2$, the list of f's Fourier coefficients $b(n)$ begins: 0, $\sqrt{3}$, 0, $\overline{\zeta}\sqrt{7}$, $\overline{\zeta}\sqrt{11}$, 0, 0, 0, 0, $-\sqrt{77}$, $-\zeta\sqrt{33}$, 0, $-\zeta\sqrt{21}$, ...

Table 1: The newform in Example 3.5

$s(p)$	E	F	$b(p)$
1,1,1	1	3003	$-iu\chi(u)$
1,1,-1	7	429	$-\zeta u\chi(u)\sqrt{7}$
1,-1,1	33	91	$-\zeta u\chi(u)\sqrt{33}$
1,-1,-1	231	13	$-u\chi(u)\sqrt{231}$
-1,1,1	11	273	$-\zeta u\chi(u)\sqrt{11}$
-1,1,-1	77	39	$-u\chi(u)\sqrt{77}$
-1,-1,1	3	1001	$-iu\chi(u)\sqrt{3}$
-1,-1,-1	21	143	$-\zeta u\chi(u)\sqrt{21}$

4 Some non-CM newforms

This section offers examples of newforms which, although not attached to Hecke characters, appear to have Fourier coefficients satisfying an analogue of (1.10). For simplicity, we restrict our examples to those with weight $k \in \{2,3,4\}$, level N divisible by exactly three primes $\ell_1 < \ell_2 < \ell_3$, and nebentypus χ of conductor

$$c = |d_1 d_2 d_3|, \tag{4.1}$$

where d_i is the prime discriminant corresponding to the prime ℓ_i, $i = 1, 2, 3$. Table 2 gives 17 examples of such weight k, level N newforms with quadratic nebentypus

$$\chi(\cdot) = \left(\frac{d_1 d_2 d_3}{\cdot}\right) \tag{4.2}$$

of conductor c, whose Fourier coefficients $b(p)$ for primes p not dividing the level have the form

$$b(p) = m(p)\sqrt{\chi(p)w(p)}, \text{ if } p \nmid N, p < 5000, \tag{4.3}$$

where $m(p)$ and $w(p)$ are nonnegative integers such that $w(p)$ depends only on the signature

$$S(p) := \left(\left(\frac{d_1}{p}\right), \left(\frac{d_2}{p}\right), \left(\frac{d_3}{p}\right)\right). \tag{4.4}$$

We conjecture that Table 2 could be extended to give infinitely many such examples. Table 2 uses the notation $w(p) = w_i$, where $i = 1, 2, 3, 4, 5, 6, 7, 8$ according as $S(p)$ equals $(1, 1, 1)$, $(-1, 1, 1)$, $(1, 1, -1)$, $(-1, 1, -1)$, $(1, -1, 1)$, $(-1, -1, 1)$, $(1, -1, -1)$, $(-1, -1, -1)$. We conjecture that the data in each row of Table 2 is valid for *all* primes $p \nmid N$, i.e., that the restriction $p < 5000$ in (4.3) can be dropped.

Table 2: 17 newforms not attached to Hecke characters

$\boldsymbol{k}$	$\boldsymbol{N}$	$\boldsymbol{c}$	$\boldsymbol{w_1}$	$\boldsymbol{w_2}$	$\boldsymbol{w_3}$	$\boldsymbol{w_4}$	$\boldsymbol{w_5}$	$\boldsymbol{w_6}$	$\boldsymbol{w_7}$	$\boldsymbol{w_8}$
2	180	60	4	60	6	10	2	30	12	20
4	300	60	4	60	100	60	80	1200	80	48
3	168	168	4	28	24	168	32	56	12	84
3	260	260	4	2496	48	52	624	4	52	48
3	336	84	4	12	66	22	2772	924	42	126
3	504	168	4	4	96	96	8	8	48	48
3	600	120	4	28	4	28	224	8	224	8
3	672	168	4	28	24	168	32	56	12	84
3	975	195	4	800	160	20	4	800	160	20
3	987	987	4	20	30	6	6	30	20	4
3	1008	84	4	84	48	112	18	42	24	2016
3	1197	399	4	8	10	20	684	38	190	3420
3	1232	308	1	115	690	6	69	15	10	46
3	1480	1480	4	348	58	6	6	58	348	4
3	1488	372	1	7	203	29	29	203	7	1
3	1508	1508	4	880	440	2	40	22	44	20
3	1584	132	4	220	10	22	2	110	20	44

The 17 newforms in Table 2 each have nebentypus of order 2, but there are similar examples with higher order nebentypus as well. For instance, consider the quartic character ψ of conductor 165 defined by

$$\psi(\cdot) = \left(\frac{33}{\cdot}\right) \chi(\cdot), \tag{4.5}$$

where χ is a quartic character $\pmod 5$ such that $\chi(2) = i$. There is a weight 3 newform for $\Gamma_0(825)$ with nebentypus ψ whose Fourier coefficients

$b(p)$ satisfy (4.3), but this time with $w(p)$ depending only on the signature

$$\left(\left(\frac{33}{p}\right), \chi(p)\right). \tag{4.6}$$

Specifically,

$$w(p) = \begin{cases} 1, & \text{if } \left(\frac{33}{p}\right) = 1, \quad \chi(p) = 1 \\ 25, & \text{if } \left(\frac{33}{p}\right) = 1, \quad \chi(p) = -1 \\ 7475, & \text{if } \left(\frac{33}{p}\right) = -1, \quad \chi(p) = 1 \\ 299, & \text{if } \left(\frac{33}{p}\right) = -1, \quad \chi(p) = -1 \\ 13, & \text{if } \left(\frac{33}{p}\right) = 1, \quad \left(\frac{5}{p}\right) = -1 \\ 23, & \text{if } \left(\frac{33}{p}\right) = -1, \quad \left(\frac{5}{p}\right) = -1. \end{cases} \tag{4.7}$$

In [3, Conj. 2.1], we gave a similar example for a weight 3 newform of level 525 with quartic nebentypus. (We associated a specific geometric object with this level 525 newform, and it would be interesting if the same could be done with the level 825 newform, as well as with the newforms in Table 2.) For the level 525 newform, we also presented in [3] conjectural congruences for the integers $m(p)$ defined in (4.3). We have similar conjectural congruences for $m(p)$ for the level 825 newform and for each of the 17 newforms in Table 2. These congruences, which depend only on the signature of p, have fascinating connections with various quadratic representations of p. We content ourselves with two examples: Conjecture 4.1 for the first newform in Table 2, and Conjecture 4.2 for the last newform in Table 2. The congruences are valid for $p < 5000$ but we conjecture that this restriction can be dropped.

Conjecture 4.1. For the weight 2, level 180 newform in Table 2, we have the following congruences for $m = m(p)$.

For $S(p) = (1, 1, 1)$,

$$m \equiv 1 \pmod 3, \tag{4.8}$$

$$\begin{cases} m \not\equiv \pm 2 \pmod 5, & \text{if } p \equiv 1 \pmod 5 \\ m \not\equiv \pm 1 \pmod 5, & \text{if } p \equiv 4 \pmod 5, \end{cases} \tag{4.9}$$

$$\begin{cases} m \equiv 0 \pmod 5, & \text{if } p = 9x^2 + 25y^2 \\ m \not\equiv 0 \pmod 5, & \text{if } p = x^2 + 225y^2. \end{cases} \tag{4.10}$$

For $S(p) = (-1, 1, 1)$,

$$m \equiv 1 \pmod 2, \text{ if } p \equiv 7 \pmod 8, \tag{4.11}$$

$$\begin{cases} m \equiv 1 \pmod 2, & \text{if } p \equiv 3 \pmod 8, \quad p = x^2 + 90y^2 \\ m \equiv 0 \pmod 2, & \text{if } p \equiv 3 \pmod 8, \quad p = 9x^2 + 10y^2. \end{cases} \tag{4.12}$$

For $S(p) = (1, 1, -1)$,

$$\begin{cases} m \equiv \pm 1 \pmod 5, & \text{if } p \equiv 3 \pmod 5 \\ m \equiv \pm 2 \pmod 5, & \text{if } p \equiv 2 \pmod 5, \end{cases} \tag{4.13}$$

$$\begin{cases} m \equiv 0 \pmod 2, & \text{if } p \equiv 1 \pmod 8 \\ m \equiv 1 \pmod 2, & \text{if } p \equiv 5 \pmod 8. \end{cases} \tag{4.14}$$

For $S(p) = (-1, 1, -1)$,

$$m \equiv 1 \pmod 3, \tag{4.15}$$

$$\begin{cases} m \equiv 0 \pmod 2, & \text{if } p \equiv 3 \pmod 8 \\ m \equiv 1 \pmod 2, & \text{if } p \equiv 7 \pmod 8. \end{cases} \tag{4.16}$$

For $S(p) = (1, -1, 1)$,

$$m \equiv 2 \pmod 3, \tag{4.17}$$

$$\begin{cases} m \equiv 0 \pmod 2, & \text{if } p \equiv 5 \pmod 8 \\ m \equiv 1 \pmod 2, & \text{if } p \equiv 1 \pmod 8, \end{cases} \tag{4.18}$$

$$\begin{cases} m \equiv \pm 1 \pmod 5, & \text{if } p \equiv 1 \pmod 5 \\ m \equiv \pm 2 \pmod 5, & \text{if } p \equiv 4 \pmod 5. \end{cases} \tag{4.19}$$

For $S(p) = (-1, -1, 1)$,

$$\begin{cases} m \equiv 0 \pmod 2, & \text{if } p \equiv 7 \pmod 8 \\ m \equiv 1 \pmod 2, & \text{if } p \equiv 3 \pmod 8. \end{cases} \tag{4.20}$$

For $S(p) = (1, -1, -1)$,

$$m \equiv 0 \pmod 2, \text{ if } p \equiv 1 \pmod 8, \tag{4.21}$$

$$\begin{cases} m \equiv 1 \pmod 2, & \text{if } p \equiv 5 \pmod 8, \quad p = 5x^2 + 72y^2 \\ m \equiv 0 \pmod 2, & \text{if } p \equiv 5 \pmod 8, \quad p = 45x^2 + 8y^2, \end{cases} \tag{4.22}$$

$$\begin{cases} m \not\equiv \pm 2 \pmod 5, & \text{if } p \equiv 3 \pmod 5 \\ m \not\equiv \pm 1 \pmod 5, & \text{if } p \equiv 2 \pmod 5, \end{cases} \tag{4.23}$$

$$\begin{cases} m \equiv 0 \pmod 5, & \text{if } 2p = 9x^2 + 25y^2 \\ m \not\equiv 0 \pmod 5, & \text{if } 2p = x^2 + 225y^2. \end{cases} \tag{4.24}$$

For $S(p) = (-1, -1, -1)$,

$$m \equiv 1 \pmod 3, \tag{4.25}$$

$$m \equiv 0 \pmod 2, \text{ if } p \equiv 3 \pmod 8, \tag{4.26}$$

$$\begin{cases} m \equiv 1 \pmod 2, & \text{if } p \equiv 7 \pmod 8, \quad p = 5x^2 + 18y^2 \\ m \equiv 0 \pmod 2, & \text{if } p \equiv 7 \pmod 8, \quad p = 2x^2 + 45y^2. \end{cases} \tag{4.27}$$

Conjecture 4.2. For the weight 3, level 1584 newform in Table 2, we have the following congruences for $m = m(p)$.

For $S(p) = (1, 1, 1)$,

$$\begin{cases} m \equiv 1 \pmod{2}, & \text{if } p \equiv 5 \pmod{8} \\ m \equiv 0 \pmod{2}, & \text{if } p \equiv 1 \pmod{8}, \end{cases} \tag{4.28}$$

$$m \not\equiv \pm 2p, \pm 4p \pmod{11}, \tag{4.29}$$

$$\begin{cases} m \equiv 0 \pmod{5}, & \text{if } p \equiv \pm 2 \pmod{5}, \quad p = x^2 - 99y^2 \\ m \equiv \pm 2 \pmod{5}, & \text{if } p \equiv \pm 2 \pmod{5}, \quad p = 9x^2 - 11y^2, \end{cases} \tag{4.30}$$

$$\begin{cases} m \equiv 0 \pmod{5}, & \text{if } p \equiv \pm 1 \pmod{5}, \quad p = 9x^2 - 11y^2 \\ m \equiv \pm 1 \pmod{5}, & \text{if } p \equiv \pm 1 \pmod{5}, \quad p = x^2 - 99y^2. \end{cases} \tag{4.31}$$

While there are values of m in each of the three congruence classes (mod 3), we further conjecture that asymptotically half of the values of m are congruent to 2 (mod 3).

For $S(p) = (-1, 1, 1)$,

$$m \equiv 0 \pmod{2}, \text{ if } p \equiv 3 \pmod{8}, \tag{4.32}$$

$$\begin{cases} m \equiv 0 \pmod{2}, & \text{if } p \equiv 7 \pmod{8}, \quad p = x^2 + 198y^2 \\ m \equiv 1 \pmod{2}, & \text{if } p \equiv 7 \pmod{8}, \quad p = 9x^2 + 22y^2. \end{cases} \tag{4.33}$$

While there are values of m in each of the three congruence classes (mod 3), we further conjecture that asymptotically only one fifth of the values of m are congruent to 0 (mod 3).

For $S(p) = (1, 1, -1)$,

$$m \not\equiv 0, \pm p, \pm 4p \pmod{11}, \tag{4.34}$$

$$\begin{cases} m \equiv 0 \pmod 2, & \text{if } p \equiv 1 \pmod 8 \\ m \equiv 1 \pmod 2, & \text{if } p \equiv 5 \pmod 8. \end{cases} \tag{4.35}$$

For $S(p) = (-1, 1, -1)$,

$$m \equiv \pm p \pmod 5, \tag{4.36}$$

$$\begin{cases} m \equiv 0 \pmod 2, & \text{if } p \equiv 7 \pmod 8 \\ m \equiv 1 \pmod 2, & \text{if } p \equiv 3 \pmod 8. \end{cases} \tag{4.37}$$

For $S(p) = (1, -1, 1)$,

$$m \not\equiv 0, \pm 4p, \pm 5p \pmod{11}, \tag{4.38}$$

$$m \equiv \pm p \pmod 5, \text{ if } p > 5, \tag{4.39}$$

$$\begin{cases} m \equiv 0 \pmod 2, & \text{if } p \equiv 5 \pmod 8 \\ m \equiv 1 \pmod 2, & \text{if } p \equiv 1 \pmod 8. \end{cases} \tag{4.40}$$

We further conjecture that asymptotically half of the values of m are congruent to $0 \pmod 3$.

For $S(p) = (-1, -1, 1)$,

$$\begin{cases} m \equiv 0 \pmod 2, & \text{if } p \equiv 3 \pmod 8 \\ m \equiv 1 \pmod 2, & \text{if } p \equiv 7 \pmod 8. \end{cases} \tag{4.41}$$

We further conjecture that asymptotically half of the values of m are congruent to $0 \pmod 3$.

For $S(p) = (1, -1, -1)$,

$$m \not\equiv \pm p, \pm 5p \pmod{11}, \tag{4.42}$$

$$m \equiv 1 \pmod 2, \text{ if } p \equiv 1 \pmod 8, \tag{4.43}$$

$$
\begin{cases} m \equiv 0 \pmod 2, & \text{if } p \equiv 5 \pmod 8, \quad p = 11x^2 + 18y^2 \\ m \equiv 1 \pmod 2, & \text{if } p \equiv 5 \pmod 8, \quad p = 2x^2 + 99y^2. \end{cases} \tag{4.44}
$$

We further conjecture that asymptotically half of the values of m are congruent to 0 (mod 3).

For $S(p) = (-1, -1, -1)$,

$$
\begin{cases} m \equiv 0 \pmod 5, & \text{if } p \equiv \pm 2 \pmod 5, \quad p = 11x^2 - 9y^2 \\ m \equiv \pm 2 \pmod 5, & \text{if } p \equiv \pm 2 \pmod 5, \quad p = 99x^2 - y^2, \end{cases} \tag{4.45}
$$

$$
\begin{cases} m \equiv 0 \pmod 5, & \text{if } p \equiv \pm 1 \pmod 5, \quad p = 99x^2 - y^2 \\ m \equiv \pm 1 \pmod 5, & \text{if } p \equiv \pm 1 \pmod 5, \quad p = 11x^2 - 9y^2, \end{cases} \tag{4.46}
$$

$$
m \equiv 0 \pmod 2, \text{ if } p \equiv 7 \pmod 8, \tag{4.47}
$$

$$
\begin{cases} m \equiv 0 \pmod 2, & \text{if } p \equiv 3 \pmod 8, \quad p = 8x^2 + 99y^2 \\ m \equiv 1 \pmod 2, & \text{if } p \equiv 3 \pmod 8, \quad p = 11x^2 + 72y^2. \end{cases} \tag{4.48}
$$

We further conjecture that asymptotically half of the values of m are congruent to 0 (mod 3).

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