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Theta Function Identities

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1. INTRODUCTION

By 1986, all but one of the identities in the 21 chapters of Ramanujan's Second Notebook [10] had been proved; see Berndt's books [2–4]. The remaining identity, which we will prove in Theorem 5.1 below, is [10, Chap. 20, Entry 8(i)]

$$\frac{1}{G_1(z)\,G_5(z)}+\frac{1}{G_2(z)\,G_3(z)}+\frac{1}{G_4(z)\,G_6(z)}=4+\frac{\eta^2(z/p)}{\eta^2(z)}, \tag{1.1}$$

where $\eta(z)$ is the classical eta function given by (2.5) and

$$G_m(z)=G_{m,p}(z)=(-1)^m\,q^{m(3m-p)/(2p^2)}\frac{f(-q^{2m/p},\,-q^{1-2m/p})}{f(-q^{m/p},\,-q^{1-m/p})}, \tag{1.2}$$

with $q=\exp(2\pi iz)$, $p=13$, and

$$f(\alpha,\beta)=\sum_{k=-\infty}^{\infty}\alpha^{(k^2+k)/2}\beta^{(k^2-k)/2}. \tag{1.3}$$

The author is grateful to Bruce Berndt for bringing (1.1) to his attention.

The quotients $G_m(z)$ in (1.2) for odd p have been the subject of interesting investigations by Ramanujan and others. Ramanujan [11, p. 207] explicitly wrote down a version of the famous quintuple product identity,

$$\frac{f(-q^2,\,-\lambda q)}{f(-q,\,-\lambda q^2)}=\frac{f(-\lambda^2q^3,\,-\lambda q^6)+qf(-\lambda,\,-\lambda^2q^9)}{f(-\lambda q^3,\,-\lambda^2q^6)}, \tag{1.4}$$

which yields as a special case a formula for $\eta(z)\,G_m(z)$ as a linear combination of two theta functions; see (1.7). In Chapter 16 of his Second

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Notebook, Ramanujan recorded the famous Rogers–Ramanujan continued fraction formula

$$q^{-1/5}G_2(z)=\frac{1}{1+}\frac{q}{1+}\frac{q^2}{1+}\frac{q^3}{1+}\cdots, \tag{1.5}$$

where $q=\exp(2\pi iz)$, $p=5$; see [1, p. 74]. Moreover, Chapters 19–20 of Ramanujan's Second Notebook contain several interesting identities involving $G_m(z)$ for odd values of p ranging from 5 through 17. K. G. Ramanathan [9] has generalized some of these identities and has investigated the signs of the Fourier coefficients of $G_m(z)$ for all odd $p \not\equiv 0 \pmod 3$.

Our paper focusses on identities for the functions $G_m(z)$. We now discuss the contents of the succeeding sections.

Sections 2 and 3 are devoted to preliminary results. Instead of expressing $G_m(z)$ as a quotient of the theta functions f defined in (1.3), we express $G_m(z)$ in Section 2 as the quotient

$$G_m(z)=(-1)^m\, F(2m/p,0;z)/F(m/p,0;z), \tag{1.6}$$

where $F(u,v;z)$ is the theta function whose series and product representations are given in (2.10). This reformulation is expedient because of the beautiful transformation formula (3.8) enjoyed by $F(u,v;z)$. In view of (1.6) and Lemma 2.1, one can also express $G_m(z)$ in the form

$$G_m(z)=\frac{(-1)^{m+1}\,i}{\eta(z)}\,\{F(1/3+m/p,0;3z)+F(1/3-m/p,0;3z)\}. \tag{1.7}$$

Lemma 2.1 is equivalent to the quintuple product formula (1.4), and we provide a short analytic proof. (For other proofs, consult [6].) Our chief application of Lemma 2.1 is to Theorem 6.1.

In Section 3, we summarize the properties of modular forms and groups that will be needed in the sequel. Lemma 3.1 states that the complex scalar multiples of $\eta^2(z)$ are the only cusp forms of weight 1 on the congruence subgroup $\Gamma(12)$ with constant multiplier. Lemma 3.1 will be used in the proof of Theorem 6.2. We make an incidental conjecture at the end of Section 3, namely, that the scalar multiples of $\eta^2(12z)$ are the only cusp forms of weight 1 on $\Gamma_1(144)$ (defined in (3.14)) with constant multiplier.

We begin Section 4 by proving Theorem 4.1, one of our main results. This provides, for each odd $p>1$, a class of modular functions $g(z)$ on $\Gamma^0(p)$ with the property that $g(z)$ has no poles on the upper half plane or at the cusp 0. Moreover, the nonconstant terms in the Fourier expansions of $g(z)$ at the cusps 0 and ∞ are rational integers (see the remark preceding Corollary 4.2). The functions $g(z)$ are constructed by summing, over all m

(mod p), products of (positive or negative) powers of $G_{m\beta}(z)$ over certain integers β. Examples of such constructions are given in Corollaries 4.2, 4.3, and 4.4.

In Section 5, Corollaries 4.2–4.4 are applied to prove Ramanujan's outstanding identity (1.1) and related identities of the type (5.1) found in [10, Chap. 20, Entry 8(i); Chap. 19, Entry 18(i)]; see Theorems 5.1 and 5.2. For an application of Theorem 5.2, see [5, p. 312]. A recipe for constructing and proving a host of formulas of this type for prime p is presented at the beginning of Section 5. The procedure, based on Theorem 4.1, is quite simple, because the function $g(z)$ constructed in Theorem 4.1 behaves nicely at the cusp 0, and ∞ is the only other cusp for $\Gamma^0(p)$ when p is prime.

Ramanujan stated four interesting identities for $p=13$ in [10, Chap. 20, Entry 8(i)], all of the type (5.1). Two of these are given in Theorem 5.1. We have been unable to generalize these two. We have, however, been able to extend the other two identities to hold for infinitely many odd p. These generalizations are given in Theorems 6.1 and 6.2. Ramanujan has given the special cases $p=5$, 7, 9, 11, 13, and 17 of Theorem 6.1 [10, Chap. 19, Entries 12(v), 17(v); Chap. 20, Entries 2(vii), 6(iii), 8(i), 12(i)] and the cases $p=13, 17$ of Theorem 6.2 [10, Chap. 20, Entries 8(i), 12(i)]. K. G. Ramanathan [9, Theorems 1, 1′] has independently proved Theorem 6.1 in the cases $p \equiv \pm 1 \pmod 6$. Our proof of Theorem 6.1 uses the quintuple product identity and our proof of Theorem 6.2 employs basic properties of Hecke operators on $\Gamma(12)$.

Two further theta function identities are given in Theorems 7.1 and 7.2. These do not seem to have been stated by Ramanujan, although they can be derived from his work. Theorem 7.1 states that, for $p=13$,

$$\frac{1}{G_1(z)\,G_5(z)}+G_4(z)\,G_6(z)=1. \tag{1.8}$$

This is equivalent to an intriguing formula involving infinite products of the form

$$(x)_\infty=\prod_{m=0}^{\infty}(1-xq^m), \tag{1.9}$$

namely,

$$\begin{aligned}\{(t^2)_\infty\,(t^3)_\infty\,(t^{10})_\infty\,(t^{11})_\infty\}^{-1}+t\{(t^4)_\infty\,(t^6)_\infty\,(t^7)_\infty\,(t^9)_\infty\}^{-1}\\ =\{(t)_\infty\,(t^5)_\infty\,(t^8)_\infty\,(t^{12})_\infty\}^{-1},\end{aligned} \tag{1.10}$$

where $t=q^{1/13}$. It would be of interest to have similar elegant formulas for values of p besides 13.

In Theorem 7.2, we provide an example of an identity of type (5.1) for a composite value of p, namely $p=9$. We close Section 7 with a new proof based on Theorem 6.1 of a result of Ramanujan, (7.37). Our proof illustrates how the results of this paper can be applied to prove certain theta function identities not precisely of type (5.1).

2. Theta and Eta Functions

Let H denote the complex upper half plane, i.e.,

$$H=\{z\in\mathbb{C} : \operatorname{Im} z>0\}. \tag{2.1}$$

For $z\in H$, $\gamma\in\mathbb{C}$, define the classical theta function

$$\Theta_1(\gamma, z)=\sum_{k=-\infty}^{\infty} \exp\left(\pi iz(k+\tfrac{1}{2})^2+2\pi i(k+\tfrac{1}{2})(\gamma-\tfrac{1}{2})\right). \tag{2.2}$$

By the Jacobi triple product formula,

$$\Theta_1(\gamma, z)=-ie^{\pi i(\gamma+z/4)}\prod_{m=1}^{\infty}(1-e^{2\pi i(\gamma+mz)}) \times(1-e^{2\pi i(-\gamma+(m-1)z)})(1-e^{2\pi imz}). \tag{2.3}$$

The classical eta function $\eta(z)$ is a simple multiple of a theta function, namely

$$\eta(z)=-ie^{\pi iz/3}\Theta_1(z, 3z), \qquad z\in H. \tag{2.4}$$

By (2.2)–(2.4),

$$\eta(z)=q^{1/24}\sum_{k=-\infty}^{\infty}(-1)^k q^{k(3k-1)/2}=q^{1/24}\prod_{m=1}^{\infty}(1-q^m), \tag{2.5}$$

where

$$q=e^{2\pi iz}. \tag{2.6}$$

So that we may eventually relate Θ_1 to modular forms in z of arbitrary level, define as in [14, Eq. (10)], for $u, v\in\mathbb{C}, z\in H$,

$$\phi(u, v; z)=e^{\pi iu(uz+v)}\Theta_1(uz+v, z)/\eta(z). \tag{2.7}$$

The function ϕ is analytic for $z\in H$, $u\in\mathbb{C}$, and $v\in\mathbb{C}$, in each variable. Write

$$F(u, v; z)=\eta(z)\,\phi(u, v; z) \tag{2.8}$$

and, when $v=0$,

$$F(u; z)=F(u, 0; z)=\eta(z)\,\phi(u, 0; z). \tag{2.9}$$

Combining (2.2), (2.3), (2.7), and (2.8), we have

$$\begin{aligned} F(u, v; z) &= e^{\pi iu(uz+v)}\Theta_1(uz+v, z) \\ &= -i \sum_{k=-\infty}^{\infty} (-1)^k \exp(i\pi z(k+u+\tfrac{1}{2})^2 + i\pi v(2k+u+1)) \\ &= -ie^{\pi i(z(u+1/2)^2+v(u+1))} \prod_{m=1}^{\infty} (1-e^{2\pi iv}q^{m+u}) \\ &\quad \times (1-e^{-2\pi iv}q^{m-1-u})(1-q^m). \end{aligned} \tag{2.10}$$

In particular, when $v=0$,

$$\begin{aligned} F(u; z) &= e^{\pi iu^2z}\Theta_1(uz, z) = -i \sum_{k=-\infty}^{\infty} (-1)^k\, q^{(k+u+1/2)^2/2} \\ &= -iq^{(u+1/2)^2/2} \prod_{m=1}^{\infty} (1-q^{m+u})(1-q^{m-1-u})(1-q^m). \end{aligned} \tag{2.11}$$

From the series in (2.10), it is easily seen that for integers r, s,

$$F(u+r, v+s; z)=(-e^{\pi iu})^s\,(-e^{-\pi iv})^r\,F(u, v; z), \tag{2.12}$$

and

$$F(-u, -v; z)=-F(u, v; z). \tag{2.13}$$

In particular, when $v=0$,

$$F(u+1; z)=-F(u; z) \tag{2.14}$$

and

$$F(-u; z)=-F(u; z). \tag{2.15}$$

By (2.11), for fixed $z\in H$, the zeros of $F(u, z)$ are the points u in the lattice $\mathbb{Z}+\mathbb{Z}z^{-1}$, and these zeros are simple. Thus $F(2u; z)/F(u; z)$ is an entire function of u. The following lemma shows in fact that $F(2u; z)/F(u; z)$ is a linear combination of $F(1/3+u; 3z)$ and $F(1/3-u; 3z)$.

LEMMA 2.1. *For $z\in H$, $u\in\mathbb{C}$,*

$$i\eta(z)\,\frac{F(2u; z)}{F(u; z)}=F(1/3+u; 3z)+F(1/3-u; 3z). \tag{2.16}$$

Proof. Replace the functions in (2.16) by their respective triple products in (2.5) and (2.11), and simplify. Then (2.16) becomes, with $a = q^u$,

$$\prod_{m=1}^{\infty} (1 + aq^m)(1 + a^{-1}q^{m-1})(1 - a^2q^{2m-1})(1 - a^{-2}q^{2m-1})(1 - q^m)$$

$$= a^{-1} \prod_{m=1}^{\infty} (1 - a^3q^{3m-2})(1 - a^{-3}q^{3m-1})(1 - q^{3m})$$

$$+ \prod_{m=1}^{\infty} (1 - a^3q^{3m-1})(1 - a^{-3}q^{3m-2})(1 - q^{3m}). \tag{2.17}$$

This is a well-known version of the quintuple product identity [6]. However, the following short analytic proof of (2.17) may be worth including.

Let $L(u)$, $R(u)$ denote the left and right members of (2.17), respectively. These are entire functions of u which satisfy, for $q = e^{2\pi i z}$,

$$L(u + 1) = -q^{-3u-2}L(u), \tag{2.18}$$

$$R(u + 1) = -q^{-3u-2}R(u), \tag{2.19}$$

$$L(u + z^{-1}) = L(u) \tag{2.20}$$

and

$$R(u + z^{-1}) = R(u). \tag{2.21}$$

The zeros of $L(u)$ are at the points

$$u = m/2 + nz^{-1}/2 \qquad (m, n \in \mathbb{Z}, \text{not both even}), \tag{2.22}$$

and these zeros are all simple. It is easily checked that

$$R(z^{-1}/2) = R(1/2) = R(1/2 + z^{-1}/2) = 0, \tag{2.23}$$

so by (2.19) and (2.21), $R(u) = 0$ at the points in (2.22). Thus

$$Q(u) := R(u)/L(u) \tag{2.24}$$

is entire. By (2.18)–(2.21),

$$Q(u + 1) = Q(u) = Q(u + z^{-1}). \tag{2.25}$$

Because of the double periodicity in (2.25), $Q(u)$ is a bounded entire function, so $Q(u)$ is constant. Finally, $Q(u) \equiv 1$, since

$$L(0) = R(0) = 2 \prod_{m=1}^{\infty} (1 - q^m). \tag{2.26}$$

This paper will focus on the quotients $F(2u; z)/F(u; z)$ in the case where u is the rational number m/p, with p odd. Thus, for integers m, p with p odd >1, define (cf. (2.36))

$$G(m; z) = (-1)^m F(2m/p; z)/F(m/p; z). \tag{2.27}$$

By (2.14), for fixed p and z, $G(m; z)$ depends only on the class of m (mod p), since p is odd. By (2.15),

$$G(m; z) = G(-m; z) = G(p-m; z). \tag{2.28}$$

By the product formula in (2.11), $G(0; z) = 2$, so

$$G(m; z) = 2, \qquad \text{if} \quad p \mid m. \tag{2.29}$$

Ramanujan worked extensively with the theta function which he denoted by

$$f(\alpha, \beta) = \sum_{k=-\infty}^{\infty} a^{(k^2+k)/2} \beta^{(k^2-k)/2}. \tag{2.30}$$

In order to relate his notation to ours, make the change of variables

$$\beta = -e^{-2\pi i \gamma}, \qquad \alpha = q/\beta, \tag{2.31}$$

where $q = e^{2\pi i z}$. Then by (2.2), (2.3), and (2.30),

$$f(\alpha, \beta) = ie^{-\pi i(\gamma + z/4)} \Theta_1(\gamma, z) = \prod_{m=1}^{\infty} (1 + \beta^{-1} q^m)(1 + \beta q^{m-1})(1 - q^m). \tag{2.32}$$

By (2.11), (2.15), and (2.32),

$$f(-q^u, -q^{1-u}) = -iq^{-(u-1/2)^2/2} F(u; z). \tag{2.33}$$

Thus,

$$\frac{F(2u; z)}{F(u; z)} = q^{u(3u-1)/2} \frac{f(-q^{2u}, -q^{1-2u})}{f(-q^u, -q^{1-u})}, \tag{2.34}$$

so

$$G(m; pz) = (-1)^m q^{m(3m-p)/(2p)} \frac{f(-q^{2m}, -q^{p-2m})}{f(-q^m, -q^{p-m})}. \tag{2.35}$$

This shows that

$$G_m(z) = G(m; z), \tag{2.36}$$

where $G_m(z)$ is given by (1.2) and $G(m; z)$ is given by (2.27).

3. Modular Transforms of Eta and Theta Functions

Define the modular group

$$\Gamma(1)=\left\{\begin{pmatrix} a & b \\ c & d \end{pmatrix}: a, b, c, d \in \mathbb{Z}, ad-bc=1\right\}. \tag{3.1}$$

Let $k\in\mathbb{R}$ and let Γ be a subgroup of $\Gamma(1)$ of finite index. Let $V: \Gamma \to \{w\in\mathbb{C}: |w|=1\}$. The space $M(\Gamma, k, V)$ of modular forms consists of those functions $g: H\to \mathbb{C}\cup\{\infty\}$ which are meromorphic on H and at every cusp, and which satisfy

$$g(Az)=V(A)(cz+d)^k g(z) \qquad \left(z\in H, A=\begin{pmatrix} a & b \\ c & d \end{pmatrix}\in\Gamma\right) \tag{3.2}$$

(so V is a multiplier system of weight k on Γ). To say that g is meromorphic at a cusp $L\infty$ $(L=(\begin{smallmatrix} \alpha & \beta \\ \gamma & \delta \end{smallmatrix})\in\Gamma(1))$ means that there exist $N\in\mathbb{Z}$, $\kappa\in\mathbb{R}$, $n\in\mathbb{Z}$ $(n\geqslant 1)$, and Fourier coefficients $b_m\in\mathbb{C}$ such that

$$(\gamma z+\delta)^{-k} g(Lz)=\sum_{m=N}^{\infty} b_m e^{2\pi i z(m+\kappa)/n} \tag{3.3}$$

for all $z\in H$ such that Im z is large. (It turns out that if $g\in M(\Gamma, k, V)$, one may take $\kappa=\kappa_L$ and $n=n_L$ in (3.3), where κ_L and n_L are the cusp parameter and cusp width defined in (3.17) and (3.18), respectively.)

If $g\in M(\Gamma, k, V)$ is analytic on H and if for every cusp $L\infty$, only positive powers of $e^{2\pi i z}$ occur in (3.3), then g is called a cusp form. We denote the subspace of cusp forms in $M(\Gamma, k, V)$ by $S(\Gamma, k, V)$.

It is a classical result [7, p. 51] that

$$\eta(z)\in S(\Gamma(1), 1/2, \varepsilon), \tag{3.4}$$

where for $A=(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix})\in\Gamma(1)$, $\varepsilon(A)$ is the 24th root of unity given by

$$\varepsilon(A)=\begin{cases} \left(\dfrac{d}{|c|}\right)\zeta_{24}^{bd(1-c^2)+c(a+d)-3c}, & \text{if } c \text{ is odd} \\ \left(\dfrac{c}{|d|}\right)\zeta_{24}^{ac(1-d^2)+d(b-c)+3(d-1)}, & \text{if } d \text{ is odd and either } c\geqslant 0 \text{ or } d\geqslant 0 \\ -\left(\dfrac{c}{|d|}\right)\zeta_{24}^{ac(1-d^2)+d(b-c)+3(d-1)}, & \text{if } d \text{ is odd, } c<0,\ d<0. \end{cases} \tag{3.5}$$

Here, $\zeta_m = \exp(2\pi i/m)$ and the Jacobi symbols are interpreted to be 1 when their "numerators" are 0. By (3.2) and (3.4),

$$\eta^2(Az) = \omega(A)(cz+d)\,\eta^2(z) \tag{3.6}$$

for all $A = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \Gamma(1)$, where by (3.5),

$$\begin{aligned} \omega(A) = \varepsilon^2(A) &= \zeta_{12}^{bd(1-c^2)+c(a+d)-3c}, && \text{if } c \text{ is odd} \\ &\ \zeta_{12}^{ac(1-d^2)+d(b-c)+3(d-1)}, && \text{if } d \text{ is odd.} \end{aligned} \tag{3.7}$$

By [14, Eq. (17)], (2.8), and (3.4),

$$F(u, v; Az) = \varepsilon(A)^3 \sqrt{cz+d}\, F(u_A, v_A; z) \tag{3.8}$$

for $u, v \in \mathbb{C}$, $z \in H$, $A = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \Gamma(1)$, where the row vector (u_A, v_A) is defined by

$$(u_A, v_A) = (u, v)\,A = (au + cv, bu + dv). \tag{3.9}$$

We will refer in the sequel to the following congruence subgroups of level N:

$$\Gamma(N) = \{A : a \equiv d \equiv 1, b \equiv c \equiv 0 \ (\mathrm{mod}\ N)\}, \tag{3.10}$$

$$\bar{\Gamma}(N) = \{A : a \equiv d \equiv \pm 1, b \equiv c \equiv 0 \ (\mathrm{mod}\ N)\}, \tag{3.11}$$

$$\Gamma_0(N) = \{A : N \mid c\}, \tag{3.12}$$

$$\Gamma^0(N) = \{A : N \mid b\}, \tag{3.13}$$

and

$$\Gamma_1(N) = \{A : a \equiv d \equiv 1, c \equiv 0 (\mathrm{mod}\ N)\}, \tag{3.14}$$

where $A = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \Gamma(1)$.

We will need the following elementary dimension estimate [12, Theorem 4.2.1]. If

$$|\Gamma(1) : \Gamma| = \mu < \infty \qquad \text{and} \qquad -I \in \Gamma \qquad \left(I = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}\right), \tag{3.15}$$

then

$$\dim S(\Gamma, k, V) \leqslant \max(0, k\mu/12 + 1 - \lambda'), \tag{3.16}$$

where λ' is the number of Γ-inequivalent cusps $L\infty$ $(L \in \Gamma(1))$ for which the cusp parameter $\kappa_L = 0$. Here κ_L is defined by

$$0 \leqslant \kappa_L < 1, \qquad V\left(L \begin{pmatrix} 1 & n_L \\ 0 & 1 \end{pmatrix} L^{-1}\right) = e^{2\pi i \kappa_L}, \tag{3.17}$$

where the cusp width n_L is the smallest positive integer for which

$$\begin{pmatrix} 1 & n_L \\ 0 & 1 \end{pmatrix} \in L^{-1}\Gamma L. \tag{3.18}$$

Let $L_1\infty, ..., L_\lambda\infty$ denote the Γ-inequivalent cusps $(L_i \in \Gamma(1))$. Then [12, (2.4.10)],

$$\mu = \sum_{i=1}^{\lambda} n_{L_i}. \tag{3.19}$$

In particular, if Γ is normal in $\Gamma(1)$, then all summands equal n_I and

$$\mu = \lambda n_I \qquad (\Gamma \text{ normal in } \Gamma(1)). \tag{3.20}$$

The following lemma shows that the scalar multiples of $\eta^2(z)$ are the only cusp forms of weight 1 on $\Gamma(12)$ with constant multiplier system.

LEMMA 3.1. $S(\Gamma(12), 1, 1) = \mathbb{C}\eta^2(z)$.

Proof. By (3.4), (3.6), and (3.7),

$$\eta^2(z) \in S(\Gamma(12), 1, 1).$$

It remains to show dim $S(\Gamma(12), 1, 1) \leqslant 1$. By (3.16) with $\Gamma = \bar{\Gamma}(12)$, $k = 1$, and $V((\begin{smallmatrix} a & b \\ c & d \end{smallmatrix})) = \chi(d)$ for any odd character $\chi(\text{mod } N)$,

$$\dim S(\Gamma(12), 1, 1) = \dim S(\bar{\Gamma}(12), 1, V) \leqslant 1 + \mu/12 - \lambda'.$$

It suffices to show $\mu/12 = \lambda'$. By (3.17), all cusp parameters κ_L vanish, so $\lambda = \lambda'$. Finally, $\lambda = \mu/12$ by (3.20).

We are grateful to R. Rankin for pointing out that a similar argument shows that

$$\dim S(\Gamma_0(144), 1, \chi_{-4}) \leqslant 1, \tag{3.21}$$

where χ_{-4} is a character (mod 144) given by

$$\chi_{-4}(d) = (-1)^{(d-1)/2}, \qquad d \text{ odd}. \tag{3.22}$$

Rankin argues as follows. By (3.18) with $\Gamma = \Gamma_0(144)$ and $L = (\begin{smallmatrix} a & b \\ c & d \end{smallmatrix})$,

$$L\begin{pmatrix} 1 & n_L \\ 0 & 1 \end{pmatrix} L^{-1} = \begin{pmatrix} 1 - acn & a^2 n \\ -c^2 n & 1 + acn \end{pmatrix} \in \Gamma_0(144).$$

Thus $144 \mid c^2 n$, so $4 \mid cn$. Consequently, $\chi_{-4}(1 + acn) = 1$, so by (3.17) with $V((\begin{smallmatrix} a & b \\ c & d \end{smallmatrix})) = \chi_{-4}(d)$, $\kappa_L = 0$ for all $L \in \Gamma(1)$. Therefore $\lambda = \lambda'$. By [13, p. 102, (33)], $\lambda=24$. By [12, (1.4.23)], $\mu=288$. Thus $\mu/12=\lambda'$, so (3.21) follows from (3.16).

In fact, equality holds in (3.21), since

$$S(\Gamma_0(144), 1, \chi_{-4}) = \mathbb{C}\eta^2(12z). \tag{3.23}$$

This follows from (3.4), (3.6), and (3.7), because for $A = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \Gamma_0(144)$,

$$\begin{aligned} \eta^2(12Az) &= \omega\left(\begin{pmatrix} a & 12b \\ c/12 & d \end{pmatrix}\right)\left(\frac{c}{12}(12z)+d\right)\eta^2(12z) \\ &= \chi_{-4}(d)(cz+d)\,\eta^2(12z). \end{aligned}$$

I conjecture that

$$\dim S(\Gamma_0(144), 1, \chi) = 0$$

for all characters $\chi(\mathrm{mod}\ 144)$ except χ_{-4}. Since [12, Theorem 8.1.1] $S(\Gamma_1(144), 1, 1)$ is the direct sum of the subspaces $S(\Gamma_0(144), 1, \chi)$ over all characters $\chi(\mathrm{mod}\ 144)$, this conjecture is equivalent to the following.

Conjecture. $S(\Gamma_1(144), 1, 1) = \mathbb{C}\eta^2(12z)$.

4. Construction of Modular Functions of Odd Level from Theta Functions

THEOREM 4.1. *Let p be odd >1 and let ε_r, β_r be nonzero integers $(1 \leqslant r \leqslant s)$ with*

$$\varepsilon_1\beta_1^2 + \cdots + \varepsilon_s\beta_s^2 \equiv 0 (\mathrm{mod}\ p). \tag{4.1}$$

Then

$$g(z) := \sum_m \prod_{r=1}^{s} G(m\beta_r; z)^{\varepsilon_r} \in M(\Gamma^0(p), 0, 1), \tag{4.2}$$

where the sum is over all m (mod p). Moreover, $g(z)$ has no poles on H or at the cusp 0.

Proof. Let

$$A = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma^0(p). \tag{4.3}$$

We first prove that g transforms like a modular function in $M(\Gamma^0(p), 0, 1)$, i.e.,

$$g(Az) = g(z), \qquad z \in H. \tag{4.4}$$

By (4.2), (2.27), and (2.9),

$$g(Az)=\sum_m \prod_r \left\{(-1)^{m\beta_r} \frac{F(2m\beta_r/p, 0; Az)}{F(m\beta_r/p, 0; Az)}\right\}^{\varepsilon_r}. \tag{4.5}$$

The expression in braces in (4.5) is to be interpreted as 2 when $p \mid m\beta_r$; see (2.29). Apply the transformation in (3.8) to obtain

$$g(Az)=\sum_m \prod_r \left\{(-1)^{m\beta_r} \frac{F(2m\beta_r a/p, 2m\beta_r b/p; z)}{F(m\beta_r a/p, m\beta_r b/p; z)}\right\}^{\varepsilon_r}. \tag{4.6}$$

Since $p \mid b$ by (4.3), $m\beta_r b/p \in \mathbb{Z}$. By (2.12), for $v \in \mathbb{Z}$,

$$F(u, v; z)=(-e^{i\pi u})^v F(u, 0; z). \tag{4.7}$$

By (4.6) and (4.7),

$$g(Az)=\sum_m \prod_r \left\{E_r(m)(-1)^{m\beta_r a} \frac{F(2m\beta_r a/p, 0; z)}{F(m\beta_r a/p, 0; z)}\right\}^{\varepsilon_r}, \tag{4.8}$$

where

$$E_r(m)=(-1)^{m\beta_r(a+1+b/p)}\, e^{3\pi i a b m^2 \beta_r^2/p^2}. \tag{4.9}$$

Rewriting (4.8) using the definition of G, we have

$$g(Az)=\sum_m \prod_r (E_r(m)\, G(m\beta_r a; z))^{\varepsilon_r}. \tag{4.10}$$

Now,

$$\prod_r E_r(m)^{\varepsilon_r}=\exp\left(i\pi m(a+1+b/p)\sum_r \varepsilon_r \beta_r+\frac{3i\pi a b m^2}{p^2}\sum_r \varepsilon_r \beta_r^2\right). \tag{4.11}$$

The sums $\sum \varepsilon_r \beta_r$ and $\sum \varepsilon_r \beta_r^2$ clearly have the same parity, and the latter sum is a multiple of p, by (4.1). Thus, if a is odd, the right side of (4.11) equals 1. If a is even, then b is odd because $ad-bc=1$, so again we see that the right side of (4.11) equals 1. Therefore (4.10) becomes

$$g(Az)=\sum_m \prod_r G(m\beta_r a; z)^{\varepsilon_r}. \tag{4.12}$$

Since $ad-bc=1$ and $p \mid b$, a is relatively prime to p. Thus am runs through a complete residue system $(\text{mod}\, p)$ when m does, so (4.4) follows from (4.12).

If $m/p \in \mathbb{Z}$, then $G(m; z)=2$ for all $z \in H$, by (2.29). If $m/p \notin \mathbb{Z}$, then since m/p is not half of an integer, both $F(2m/p; z)$ and $F(m/p; z)$ are analytic functions of z on H which never vanish on H, by (2.11). Thus $G(m; z)^{\varepsilon}$ is analytic on H for all $m, \varepsilon \in \mathbb{Z}$, so $g(z)$ is analytic on H. It remains to show that $g(z)$ is meromorphic at every cusp $L\infty$ ($L \in \Gamma(1)$) with no pole at the cusp 0.

By (2.27) and (3.8), for any $m \in \mathbb{Z}$ and $L = \left(\begin{smallmatrix} \alpha & \beta \\ \gamma & \delta \end{smallmatrix}\right) \in \Gamma(1)$,

$$G(m; Lz) = (-1)^m F(2m\alpha/p, 2m\beta/p; z)/F(m\alpha/p, m\beta/p; z). \tag{4.13}$$

(The right side of (4.13) is interpreted as 2 if $p \mid m$.) By (2.10) and (4.13), we see that for any $m, \varepsilon \in \mathbb{Z}$, $G(m; Lz)^{\varepsilon}$ has a Fourier expansion of the form

$$G(m; Lz)^{\varepsilon} = \sum_{k=N}^{\infty} a_k e^{2\pi i z k/p}, \qquad a_k \in \mathbb{C}, \tag{4.14}$$

where N is finite. Thus $g(Lz)$ also has a Fourier expansion of this form, so $g(z)$ is meromorphic at every cusp. This completes the proof that $g(z) \in M(\Gamma^0(p), 0, 1)$, and it remains to show that $g(z)$ has no pole at the cusp 0. This will be accomplished by showing that for each $m \in \mathbb{Z}$, $G_m(-1/z)$ has a Fourier expansion of the form

$$G_m(-1/z) = \sum_{k=0}^{\infty} c_k q^k, \tag{4.15}$$

where

$$c_0 = 2(-1)^m \cos(\pi m/p) \neq 0. \tag{4.16}$$

If $p \mid m$, then (4.15) holds since then $G_m(-1/z) = 2$. Let $p \nmid m$. By (4.13),

$$G_m(-1/z) = (-1)^m F(0, -2m/p; z)/F(0, -m/p; z). \tag{4.17}$$

By (2.10),

$$\begin{aligned} q^{-1/8} F(0, -m/p; z) &= -i \sum_{k=-\infty}^{\infty} (-1)^k q^{(k^2+k)/2} e^{-\pi i(2k+1)m/p} \\ &= -2 \sum_{k=0}^{\infty} (-1)^k \sin\left(\frac{\pi m(2k+1)}{p}\right) q^{(k^2+k)/2}. \end{aligned} \tag{4.18}$$

Now (4.15)–(4.16) follow from (4.17)–(4.18).

Remark. Let $\sigma = \varepsilon_1 + \cdots + \varepsilon_s$. We claim that the Fourier expansions of both $g(z) - 2^{\sigma}$ and $g(-1/z) - 2^{\sigma}$ have integral coefficients. (The term 2^{σ} corresponds to the term for $m = 0$ in (4.2).) To see this, first note that if $p \nmid m$, then by (2.11), $iF(m/p; z)$ has (Fourier) coefficients ± 1, so by (2.27),

$G_m(z)^\varepsilon$ has integral coefficients for any $\varepsilon \in \mathbb{Z}$. Hence $g(z) - 2^\sigma$ has integral coefficients, by (4.2). Next note that if $p \nmid m$, then by (4.18),

$$q^{-1/8} F(0, -m/p; z) = \left(-2 \sin \frac{\pi m}{p}\right) \sum_{k=0}^{\infty} (-1)^k q^{(k^2+k)/2} \left\{\frac{\sin(\pi m(2k+1)/p)}{\sin(\pi m/p)}\right\},$$

and the quotients in baces are in the ring of cyclotomic integers $\mathbb{Z}[\zeta_p]$. Since $\sin(2\pi m/p)/\sin(\pi m/p)$ is a unit in $\mathbb{Z}[\zeta_p]$, if follows from (4.17) that $G_m(-1/z)^\varepsilon$ has coefficients in $\mathbb{Z}[\zeta_p]$ for any $\varepsilon \in \mathbb{Z}$. Because of the way these coefficients depend on m and because

$$\sum_{m (\mathrm{mod}\ p)} \zeta_p^{mn} \in \mathbb{Z} \qquad \text{for all} \quad n \in \mathbb{Z},$$

it follows from (4.2) that $g(-1/z) - 2^\sigma$ has integral coefficients.

COROLLARY 4.2. *Let $p > 5$ be a prime $\equiv 1 \pmod 4$ and let $\boldsymbol{R}$ denote the set of quadratic residues* $(\mathrm{mod}\ p)$ *between* 1 *and* $p/2$. *Then*

$$h(z) := \prod_{\beta \in \boldsymbol{R}} G_\beta(z) + (-1)^{(p^2-1)/8} \prod_{\beta \in \boldsymbol{R}} G_\beta(z)^{-1} \tag{4.19}$$

is in $M(\Gamma^0(p), 0, 1)$ and has no poles on H or at the cusp 0.

Proof. In Theorem 4.1, let $s = (p-1)/4$, $\varepsilon_r = 1$ $(1 \leqslant r \leqslant s)$, and let $\beta_1, ..., \beta_s$ be the elements of $\boldsymbol{R}$. (Note that $\{\pm\beta_r : 1 \leqslant r \leqslant s\}$ is a complete set of quadratic residues $(\mathrm{mod}\ p)$.) Write $B = \beta_1^2 + \cdots + \beta_s^2$. For a primitive root g $(\mathrm{mod}\ p)$,

$$2 \sum_r \beta_r^2 + 2 \sum_r (g\beta_r)^2 \equiv \sum_{m=1}^{p-1} m^2 \equiv 0 \pmod p,$$

so $B(1+g^2) \equiv 0 (\mathrm{mod}\ p)$. Since $p > 5$, $1 + g^2 \not\equiv 0 \pmod p$, so $B \equiv 0 \pmod p$. Thus (4.1) holds. Now, for $g(z)$ as in (4.2),

$$g(z) = \sum_m \prod_{\beta \in \boldsymbol{R}} G(m\beta; z) = 2^{(p-1)/4} + \sum_{\substack{m \\ (m/p)=1}} \prod_{\beta \in \boldsymbol{R}} G_\beta(z) + \sum_{\substack{m \\ (m/p)=-1}} \prod_{\beta \in \boldsymbol{N}} G_\beta(z),$$

where $\boldsymbol{N}$ is the set of s quadratic nonresidues $(\mathrm{mod}\ p)$ between 1 and $p/2$. Therefore,

$$g(z) = 2^{(p-1)/4} + \frac{p-1}{2} \left(\prod_{\beta \in \boldsymbol{R}} G_\beta(z) + \prod_{\beta \in \boldsymbol{N}} G_\beta(z) \right). \tag{4.20}$$

By (2.27),

$$\prod_{\beta \in N} G_\beta(z) \prod_{\beta \in R} G_\beta(z) = \prod_{m=1}^{(p-1)/2} G_m(z) = (-1)^{(p^2-1)/8} \prod_{m=1}^{(p-1)/2} \frac{F(2m/p; z)}{F(m/p; z)}.$$

The rightmost product equals 1, since $F(u; z) = F(1-u; z)$ by (2.14) and (2.15). Thus (4.20) becomes

$$g(z) = 2^{(p-1)/4} + \frac{p-1}{2} h(z), \tag{4.21}$$

where $h(z)$ is defined by (4.19). The result now follows from Theorem 4.1.

COROLLARY 4.3. *Let p be a prime $\equiv 1 \pmod 4$ and let β be a primitive 4th root of unity* $(\mathrm{mod}\, p)$. *Then for $\varepsilon \in \{\pm 1\}$,*

$$k_\varepsilon(z) := \sum_{m (\mathrm{mod}\, p)} G^\varepsilon(m; z)\, G^\varepsilon(\beta m; z) \tag{4.22}$$

is in $M(\Gamma^0(p), 0, 1)$ and has no poles on H or at the cusp 0.

Proof. This follows from Theorem 4.1 with $s = 2$, $\varepsilon_1 = \varepsilon_2 = \varepsilon$, $\beta_1 = 1$, $\beta_2 = \beta$.

COROLLARY 4.4. *Let p be odd > 1. Then*

$$g(z) := \sum_{m (\mathrm{mod}\, p)} G_m(z)^p \tag{4.23}$$

is in $M(\Gamma^0(p), 0, 1)$ and has no poles on H or at the cusp 0.

Proof. This follows from Theorem 4.1 with $s = 1$, $\varepsilon_1 = p$, $\beta_1 = 1$.

5. APPLICATION TO RAMANUJAN'S IDENTITIES

For prime p, Theorem 4.1 provides a simple recipe for the creation of theta function identities of the type

$$g(z) = E(z), \tag{5.1}$$

where $g(z)$ is given by (4.2) and $E(z)$ is a relatively simple function in $M(\Gamma^0(p), 0, 1)$ composed of eta functions (as in (5.17), e.g.). The idea is to construct a function $E(z) \in M(\Gamma^0(p), 0, 1)$ with no poles except possibly at the cusp ∞ such that $g(z) - E(z)$ has a zero at ∞. Then since 0 and ∞ are the only inequivalent cusps (mod $\Gamma^0(p)$) when p is prime, it follows from

Theorem 4.1 that $g(z)-E(z)$ has no poles at all. As constants are the only entire modular functions in $M(\Gamma^0(p), 0, 1)$ [12, p. 108], (5.1) follows.

To find the Fourier espansion of $g(z)$ at ∞, we need to have the Fourier expansion of $G_m(z)$. Just as the Euler pentagonal number theorem (2.5) provides the Fourier expansion of $\eta(z)$, the quintuple product formula (Lemma 2.1) provides the Fourier expansion of $\eta(z)\, G(m; z)$. Thus, by Lemma 2.1 and (2.11),

$$\eta(z)(-1)^m\, G(m; z) = \sum_{k=-\infty}^{\infty} (-1)^k \left(q^{3(k+m/p-1/6)^2/2} + q^{3(k-m/p-1/6)^2/2}\right). \tag{5.2}$$

Since $G(m; z) = G(p-m; z)$ by (2.28), we assume

$$1 \leqslant m \leqslant (p-1)/2. \tag{5.3}$$

Isolating the terms in (5.2) for which $k=0, \pm 1$, we find that

$$\begin{aligned}\eta(z)(-1)^m\, G(m; z) &= q^{1/24} q^{(3m^2-mp)/(2p^2)} \\ &\quad \times \{1 + q^{m/p} - q^{(p-2m)/p} - q^{(p+3m)/p} - q^{(2p-3m)/p} + O(q^2)\}.\end{aligned} \tag{5.4}$$

By (2.5),

$$\eta(z) = q^{1/24}(1 - q - q^2 + O(q^5)). \tag{5.5}$$

Thus, for $1 \leqslant m \leqslant (p-1)/2$,

$$\begin{aligned}(-1)^m\, G(m; z) &= q^{(3m^2-mp)/(2p^2)} \cdot \{1 + q^{m/p} - q^{(p-2m)/p} - q^{(p+3m)/p} - q^{(2p-3m)/p} \\ &\quad + q + q^{(p+m)/p} - q^{(2p-2m)/p} - q^{(3p-3m)/p} + O(q^2)\}.\end{aligned} \tag{5.6}$$

In particular, for $1 \leqslant m \leqslant (p-1)/2$,

$$(-1)^m\, G(m; z) = q^{(3m^2-mp)/(2p^2)}\{1 + q^{m/p} - q^{(p-2m)/p} + O(q^{2/p})\}. \tag{5.7}$$

We now prove two identities of the type (5.1) stated by Ramanujan in [10, Chap. 20, Entry 8(i)]. The first identity, (5.8), is the outstanding result (1.1).

THEOREM 5.1. *Let $p = 13$. Then*

$$G_1(z)^{-1}\, G_5(z)^{-1} + G_2(z)^{-1}\, G_3(z)^{-1} + G_4(z)^{-1}\, G_6(z)^{-1} = 4 + \eta^2(z/p)/\eta^2(z), \tag{5.8}$$

and

$$G_1(z)\,G_3(z)\,G_4(z)-G_1^{-1}(z)\,G_3^{-1}(z)\,G_4^{-1}(z)=3+\eta^2(z/p)/\eta^2(z). \tag{5.9}$$

Proof. By Corollaries 4.3 and 4.2, respectively, the left members of (5.8) and (5.9) are in $M(\Gamma^0(13), 0, 1)$ and have no poles on H or at the cusp 0. We proceed to follow the recipe for proving identities of type (5.1) outlined at the beginning of this section. By (5.7), for $p=13$,

$$G_1(z)=-q^{-5/p^2}(1+q^{1/p}+O(q^{2/p})) \tag{5.10}$$

$$G_2(z)=q^{-7/p^2}(1+O(q^{2/p})) \tag{5.11}$$

$$G_3(z)=-q^{-6/p^2}(1+O(q^{2/p})) \tag{5.12}$$

$$G_4(z)=q^{-2/p^2}(1+O(q^{2/p})) \tag{5.13}$$

$$G_5(z)=-q^{5/p^2}(1+O(q^{2/p})) \tag{5.14}$$

$$G_6(z)=q^{15/p^2}(1-q^{1/p}+O(q^{2/p})). \tag{5.15}$$

For $p\equiv 1 \pmod{12}$, (3.6) and (3.7) yield

$$\eta^2\left(\frac{Az}{p}\right)=\omega\left(\begin{pmatrix} a & b/p \\ pc & d \end{pmatrix}\right)(cz+d)\,\eta^2(z/p)=\omega(A)(cz+d)\,\eta^2(z/p) \tag{5.16}$$

for $A=\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)\in\Gamma^0(p)$. Thus, for $p\equiv 1 \pmod{12}$,

$$\eta^2(z/p)/\eta^2(z)\in M(\Gamma^0(p), 0, 1). \tag{5.17}$$

Thus all members of (5.8) and (5.9) are in $M(\Gamma^0(13), 0, 1)$ and have no poles except at ∞. By (5.5),

$$\eta(z/p)/\eta(z)=q^{(1-p)/24p}\{1-q^{1/p}+O(q^{2/p})\}. \tag{5.18}$$

For $p=13$, (5.18) yields

$$\eta^2(z/p)/\eta^2(z)=q^{-1/p}-2+O(q^{1/p}). \tag{5.19}$$

From (5.19) and (5.10)–(5.15), both sides of (5.8) equal $q^{-1/p}+2+O(q^{1/p})$, while both sides of (5.9) equal $q^{-1/p}+1+O(q^{1/p})$. Thus (5.8) and (5.9) hold.

We close this section with one further identity of the type (5.1), essentially stated by Ramanujan in [10, Chap. 19, Entry 18(i)]. For an application, see [5, p. 312].

THEOREM 5.2. *Let $p = 7$. Then*

$$G_1(z)^7 + G_2(z)^7 + G_3(z)^7 = -57 - 14\left(\frac{\eta(z/7)}{\eta(z)}\right)^4 - \left(\frac{\eta(z/7)}{\eta(z)}\right)^8. \quad (5.20)$$

Proof. With

$$g(z) := \sum_{m=0}^{6} G_m(z)^7, \quad (5.21)$$

we see that (5.20) is equivalent to

$$g(z) = 14 - 28\left(\frac{\eta(z/7)}{\eta(z)}\right)^4 - 2\left(\frac{\eta(z/7)}{\eta(z)}\right)^8. \quad (5.22)$$

By Corollary 4.4, the left side of (5.22) is in $M(\Gamma^0(7), 0, 1)$ and has no poles on H or at the cusp 0. Let $E(z)$ denote the right side of (5.22). By the argument used to obtain (5.17), we have, for $p \equiv 1 \pmod 6$,

$$\eta^4(z/p)/\eta^4(z) \in M(\Gamma^0(p), 0, 1). \quad (5.23)$$

Thus both members of (5.22) are in $M(\Gamma^0(7), 0, 1)$ and have no poles except at ∞, so by the procedure described below (5.1), (5.22) will follow if

$$g(z) - E(z) \text{ has a zero at } \infty. \quad (5.24)$$

Using (5.5), we see that $E(z)$ has the Fourier expansion

$$-2q^{-2/7} - 12q^{-1/7} + 86 + O(q^{1/7}). \quad (5.25)$$

Using (5.6), we see that

$$G_1(z)^7 = -q^{-2/7}(1 + q^{1/7} + O(q^{5/7}))^7, \quad (5.26)$$

$$G_2(z)^7 = q^{-1/7}(1 + O(q^{2/7}))^7, \quad (5.27)$$

and

$$G_3(z)^7 = O(q^{3/7}). \quad (5.28)$$

Thus $g(z)$ also has a Fourier expansion as in (5.25), so (5.24) follows.

6. EXTENSIONS OF IDENTITIES OF RAMANUJAN

In Theorem 5.1, two identities of the type (5.1) were proved for $p = 13$. These are stated by Ramanujan in [10, Chap. 20, Entry 8(i)]. In this section, we prove two further identities of type (5.1) which Ramanujan also stated for $p=13$ in [10, Chap. 20, Entry 8(i)]. More importantly, we extend these identities to hold for infinitely many p.

Theorem 6.1 below is a consequence of the quintuple product identity. As was mentioned in Section 1, Ramanujan has stated the cases $p=5, 7, 9, 11, 13, 17$ of Theorem 6.1, and Ramanathan has proved the cases $p\equiv \pm 1 \pmod 6$.

THEOREM 6.1. *For any odd integer $p>1$,*

$$\sum_{m \pmod p} G(m; z) = 2(3/p)\, \eta(z/p^2)/\eta(z), \tag{6.1}$$

where $(3/p)$ is the Legendre symbol.

Proof. By (5.2),

$$\begin{aligned}\eta(z)\sum_m G(m; z) &= \sum_m \sum_{k=-\infty}^{\infty} (-1)^{k+m} \left(q^{3(k+m/p+1/6)^2/2} + q^{3(k-m/p+1/6)^2/2}\right)\\ &= 2\sum_{j=-\infty}^{\infty} (-1)^j q^{3(j/p+1/6)^2/2} = 2\sum_{j=-\infty}^{\infty} (-1)^j q^{3(j+p/6)^2/(2p^2)}.\end{aligned} \tag{6.2}$$

By (2.5),

$$\eta(z) = \sum_{k=-\infty}^{\infty} (-1)^k q^{3(k+1/6)^2/2}, \tag{6.3}$$

so it remains to show that

$$\sum_{j=-\infty}^{\infty} (-1)^j q^{3(j+p/6)^2/2} = (3/p) \sum_{k=-\infty}^{\infty} (-1)^k q^{3(k+1/6)^2/2}. \tag{6.4}$$

This is easily checked in the cases $p\equiv \pm 1 \pmod{12}$, $p\equiv \pm 5 \pmod{12}$, wherein $(3/p)=1$, $(3/p)=-1$, respectively. Finally, suppose that $3 \mid p$, so $(3/p)=0$. Then the sum on the left side of (6.4) equals

$$\pm \sum_{j=-\infty}^{\infty} (-1)^j q^{3(j+1/2)^2/2}. \tag{6.5}$$

The jth summand in (6.5) is the negative of the $(-1-j)$th summand, for $j=0, 1, 2, \ldots$, so the sum in (6.5) vanishes. Thus (6.4) holds.

In Theorem 5.1, we evaluated the function $k_\varepsilon(z)$, defined in (4.22), for $p=13$, $\varepsilon=-1$. In Theorem 6.2 below, we evaluate $k_\varepsilon(z)$ for $\varepsilon=1$ and *all* primes $p \equiv 1 \pmod 4$. Ramanujan has stated the special cases $p = 13, 17$ of Theorem 6.2[10, Chap. 20, Entries 8(i) and 12(i)].

We are very grateful to H. M. Stark for helpful suggestions relating to the proof of Theorem 6.2.

THEOREM 6.2. *For each prime* $p \equiv 1 \pmod 4$,

$$\sum_{m \,(\mathrm{mod}\, p)} G(m; z)\, G(m\beta; z) = 2a_p \eta^2(z/p)/\eta^2(z), \tag{6.6}$$

where β *is any primitive fourth root of unity* $(\mathrm{mod}\, p)$, *and*

$$a_p = \sum_{\substack{m,\, n \in \mathbb{Z} \\ (6m-1)^2 + (6n-1)^2 = 2p}} (-1)^{m+n}. \tag{6.7}$$

Proof. Let p be a prime $\equiv 1 \pmod 4$. By a general theorem on Hecke operators [12, Theorem 9.2.1], the space of cusp forms $S(\Gamma(12), 1, 1)$ is invariant under the Hecke operator T_p defined for $f \in S(\Gamma(12), 1, 1)$ by

$$f(z) | T_p := f(pz) + \frac{1}{p} \sum_{v=0}^{p-1} f\left(\frac{z+12v}{p}\right). \tag{6.8}$$

Since $S(\Gamma(12), 1, 1) = \mathbb{C}\eta^2(z)$ by Lemma 3.1, it follows that for some $\alpha_p \in \mathbb{C}$,

$$\eta^2(pz) + \frac{1}{p} \sum_{v=0}^{p-1} \eta^2\left(\frac{z+12v}{p}\right) = \alpha_p \eta^2(z). \tag{6.9}$$

Since, by (5.5), $\eta^2(z)$ has the Fourier expansion $q^{1/12}(1 - 2q + \cdots)$, comparison of the coefficients of $q^{1/12}$ in (6.9) shows that α_p is the coefficient of $q^{p/12}$ in the Fourier expansion of $\eta^2(z)$. Squaring the Fourier expansion for $\eta(z)$ given in (2.5), we thus see that α_p equals the expression a_p in (6.7).

For a modular form $h(z)$ with a Fourier expansion of the form

$$h(z) = \sum_{k \in \mathbb{Z}} b_k q^{k/(12p)}, \qquad b_k \in \mathbb{C}, \tag{6.10}$$

define

$$I(h) = \sum_{p | k} b_k q^{k/(12p)} = \frac{1}{p} \sum_{v=0}^{p-1} h(z + 12v). \tag{6.11}$$

Thus $I(h)$ is the sum of those terms of (6.10) with *integral* powers of $q^{1/12}$. Now (6.9) can be rewritten as

$$\eta^2(pz) + I(\eta^2(z/p)) = a_p \eta^2(z). \tag{6.12}$$

Squaring both sides of (6.1), we obtain

$$\eta^2(z/p) = \tfrac{1}{4} \sum_{m,\, n (\mathrm{mod}\, p)} (\eta(pz)\, G_m(pz))(\eta(pz)\, G_n(pz)). \tag{6.13}$$

By (5.2),

$$\eta(pz)\, G_m(pz) = (-1)^m q^{1/24} \sum_{k=-\infty}^{\infty} (-1)^k (q^{(pk+m)(3pk+3m-p)/(2p)} + q^{(pk-m)(3pk-3m-p)/(2p)}). \tag{6.14}$$

Thus, either all or none of the terms in the Fourier expansion of the product $(\eta(pz)\, G_m(pz))(\eta(pz)\, G_n(pz))$ will contain integral powers of $q^{1/12}$, according as m^2+n^2 is divisible by p or not. Note that m^2+n^2 is divisible by p if and only if $n \equiv \pm m\beta (\mathrm{mod}\, p)$, and then there are two such values n for each nonzero $m (\mathrm{mod}\, p)$. Thus, by (6.13), and (2.28), (2.29),

$$I(\eta^2(z/p)) = -\eta^2(pz) + \tfrac{1}{2} \sum_{m (\mathrm{mod}\, p)} (\eta(pz)\, G(m; pz))(\eta(pz)\, G(m\beta; pz)). \tag{6.15}$$

By (6.15) and (6.12),

$$\begin{aligned} \sum_{m (\mathrm{mod}\, p)} G(m; pz)\, G(m\beta; pz) &= 2(\eta^2(pz) + I(\eta^2(z/p)))/\eta^2(pz) \\ &= 2a_p \eta^2(z)/\eta^2(pz), \end{aligned}$$

and (6.6) follows.

7. Applications of Section 6

The following theorem offers an interesting identity involving infinite products of the form

$$(x)_\infty = \prod_{m=0}^{\infty} (1 - xq^m). \tag{7.1}$$

THEOREM 7.1. *For* $t = q^{1/13}$,

$$\begin{aligned} &\{(t^2)_\infty (t^3)_\infty (t^{10})_\infty (t^{11})_\infty\}^{-1} + t\{(t^4)_\infty (t^6)_\infty (t^7)_\infty (t^9)_\infty\}^{-1} \\ &\quad = \{(t)_\infty (t^5)_\infty (t^8)_\infty (t^{12})_\infty\}^{-1}. \end{aligned} \tag{7.2}$$

Equivalently, for $p = 13$,

$$G_1^{-1}(z)\, G_5^{-1}(z) + G_4(z)\, G_6(z) = 1. \tag{7.3}$$

Proof. For brevity, write

$$G_m = G_m(z). \tag{7.4}$$

With $p = 13$, $\beta = 5$, (6.6) becomes

$$4(1 + G_1 G_5 + G_2 G_3 + G_4 G_6) = -4\eta^2(z/p)/\eta^2(z), \tag{7.5}$$

so

$$G_1 G_5 + G_2 G_3 + G_4 G_6 = -1 - \eta^2(z/p)/\eta^2(z). \tag{7.6}$$

By (5.8),

$$(G_1 G_5)^{-1} + (G_2 G_3)^{-1} + (G_4 G_6)^{-1} = 4 + \eta^2(z/p)/\eta^2(z). \tag{7.7}$$

Adding (7.6) and (7.7), we obtain

$$A + A^{-1} + B + B^{-1} - AB - (AB)^{-1} \equiv 3, \tag{7.8}$$

with

$$A = G_1 G_5, \qquad B = G_4 G_6, \tag{7.9}$$

since by the definition (2.27) of G,

$$G_1 G_2 G_3 G_4 G_5 G_6 = -1. \tag{7.10}$$

We are grateful to Peter Montgomery for pointing out that (7.8) is equivalent to

$$(AB - A + 1)(AB - B + 1) \equiv 0. \tag{7.11}$$

By (5.13) and (5.15), $B \to 0$ as $q \to 0$. Thus $AB - B + 1 \not\equiv 0$, so $AB - A + 1 = 0$, i.e.,

$$B - 1 + A^{-1} = 0. \tag{7.12}$$

This proves (7.3).

By the product formula in (2.11),

$$A = \frac{F(2/p; z)\, F(3/p; z)}{F(1/p; z)\, F(5/p; z)} = \frac{(t^2)_\infty\, (t^{11})_\infty\, (t^3)_\infty\, (t^{10})_\infty}{(t^1)_\infty\, (t^{12})_\infty\, (t^5)_\infty\, (t^8)_\infty} \tag{7.13}$$

and

$$B = \frac{F(5/p; z)\, F(1/p; z)}{F(4/p; z)\, F(6/p; z)} = \frac{t(t^5)_\infty\, (t^8)_\infty\, (t)_\infty\, (t^{12})_\infty}{(t^4)_\infty\, (t^9)_\infty\, (t^6)_\infty\, (t^7)_\infty}. \tag{7.14}$$

Now (7.2) follows from (7.12)–(7.14).

In the next theorem, we essentially evaluate the modular function $g(z)$ given in (4.2), in the case $p=9$, $\varepsilon_1=\beta_1=1$, $\varepsilon_2=\beta_2=2$. The method described at the beginning of Section 5 could be used, but cusps other than 0 and ∞ would have to be considered, since $p=9$ is not prime. We base a proof instead on Theorem 6.1 and the following identities of Ramanujan proved in [4, Chap. 20, Entries 2(v), (vi), (viii)]:

$$F(4/9; z)-F(1/9; z)-F(2/9; z)=i\eta(z/27); \tag{7.15}$$

$$F(1/9; z)\, F(2/9; z)\, F(4/9; z)=-i\eta^3(z)\,\eta(z/9)/\eta(z/3); \tag{7.16}$$

$$\frac{F(4/9; z)}{F(1/9; z)}+\frac{F(2/9; z)}{F(4/9; z)}-\frac{F(1/9; z)}{F(2/9; z)}=\frac{\eta^4(z/3)}{\eta^3(z)\,\eta(z/9)}. \tag{7.17}$$

THEOREM 7.2. *For $p=9$, define*

$$h(z) := G_1(z)\, G_2(z)^2+G_2(z)\, G_4(z)^2+G_4(z)\, G_1(z)^2. \tag{7.18}$$

Then

$$h(z)=6-\frac{\eta(z/3)(3\eta^3(z/3)+\eta^3(z/27))}{\eta(z/9)\,\eta^3(z)}. \tag{7.19}$$

Proof. Let $p=9$. As in (7.4), write $G_m=G_m(z)$. By Theorem 6.1,

$$\sum_{m=0}^{8} G_m=0. \tag{7.20}$$

By (2.14), (2.15), we have $F(3/9; z)=F(6/9; z)$, so

$$G_3=G_6=-1. \tag{7.21}$$

Thus, since $G_0=2$, (7.20) yields

$$G_1+G_2+G_4=0. \tag{7.22}$$

For brevity, set

$$A=F(1/9; z), \qquad B=F(2/9; z), \qquad C=-F(4/9; z), \tag{7.23}$$

so

$$G_1=-B/A, \qquad G_2=-C/B, \qquad G_4=-A/C. \tag{7.24}$$

Then (7.22) and (7.15)–(7.18) are respectively equivalent to

$$CB^2 + AC^2 + BA^2 = 0, \tag{7.25}$$

$$A + B + C = -i\eta(z/27), \tag{7.26}$$

$$ABC = i\eta^3(z)\,\eta(z/9)/\eta(z/3), \tag{7.27}$$

$$(BC^2 + AB^2 + CA^2)/(ABC) = -\frac{\eta^4(z/3)}{\eta^3(z)\,\eta(z/9)}, \tag{7.28}$$

and

$$-h(z) = (A^3 + B^3 + C^3)/(ABC). \tag{7.29}$$

Cubing in (7.26) and then dividing by the members of (7.27), we obtain

$$\frac{\eta^3(z/27)\,\eta(z/3)}{\eta(z/9)\,\eta^3(z)} = \frac{(A+B+C)^3}{ABC} = -h(z) - \frac{3\eta^4(z/3)}{\eta^3(z)\,\eta(z/9)} + 6. \tag{7.30}$$

This proves (7.19).

Remark. Define $g(z)$ as in Theorem 4.1 for $p = 9$, $\varepsilon_1 = \beta_1 = 1$, $\varepsilon_2 = \beta_2 = 2$. In view of (7.21), the function $h(z)$ in (7.18) equals $(g(z) - 6)/2$. Thus, by Theorem 4.1, $h(z) \in M(\Gamma^0(9), 0, 1)$. Theorem 7.2 therefore implies that

$$E(z) := \frac{\eta(z/3)(3\eta^3(z/3) + \eta^3(z/27))}{\eta(z/9)\,\eta^3(z)} \in M(\Gamma^0(9), 0, 1). \tag{7.31}$$

The validity of (7.31) is not *directly* evident, although it is easy to see directly that $E(z) \in M(\Gamma^0(27), 0, 1)$. A more direct way to verify (7.31) is to use Jacobi's identity [8, p. 172],

$$\eta^3(z) = \sum_{k=0}^{\infty} (-1)^k\,(2k+1)\,q^{(2k+1)^2/8}, \tag{7.32}$$

to prove that

$$E(z+9) = E(z). \tag{7.33}$$

Then (7.31) follows since $(U^9, \Gamma^0(27)) = \Gamma^0(9)$ [12, Theorem 1.4.5], where $U = \left(\begin{smallmatrix} 1 & 1 \\ 0 & 1 \end{smallmatrix}\right)$. Note that since the Fourier expansion of $E(z)$ begins with the term $q^{-1/9}$, the level of $E(z)$ must be exactly 9.

The following product identity of Ramanujan [1, (28.1)] holds for all odd $p > 1$:

$$\prod_{m=1}^{(p-1)/2} F(m/p; z) = i^{(p-1)/2}\,\eta(z/p)\,\eta^{(p-3)/2}(z). \tag{7.34}$$

The special case (7.16) for $p=9$ was used along with Theorem 6.1 to prove Theorem 7.2. We now give one further application of Theorem 6.1. Let $p=5$. Then Theorem 6.1 yields

$$G(1; z)+G(2; z)=-1-\eta(z/25)/\eta(z) \tag{7.35}$$

and (7.34) becomes

$$F(1/5; z)\,F(2/5; z)=-\eta(z)\,\eta(z/5). \tag{7.36}$$

Multiplying, we obtain the following result of Ramanujan [1, Entry 38(iv)]:

$$F^2(1/5; z)-F^2(2/5; z)=\eta(z/5)\,\eta(z)+\eta(z/5)\,\eta(z/25). \tag{7.37}$$

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