JOURNAL OF NUMBER THEORY 54, 1-11 (1995)

Selberg–Jack Character Sums of Dimension 2

RONALD J. EVANS

Department of Mathematics, University of California at San Diego, La Jolla, California 92093-0112

Communicated by A. C. Woods

Received September 18, 1990; revised February 5, 1994

Kadell extended Selberg's n-dimensional beta integral formula by inserting a normalized Jack polynomial as a factor in the integrand. We formulate and prove a character sum analog of Kadell's formula in the case $n=2$, raising hope that such an analog may exist for general n.

1. INTRODUCTION

Kadell [15] has proved the following far-reaching extension of Selberg's n-dimensional beta integral formula [2, p. 48; 18]:

$$\int_0^1 \cdots \int_0^1 \prod_{i=1}^{n} t_i^{a-1}(1-t_i)^{b-1} \prod_{1 \leqslant i < j \leqslant n} (t_i - t_j)^{2c} s_\lambda^c(t_1, ..., t_n)\, dt_1 \cdots dt_n$$

$$= n! \prod_{j=0}^{n-1} \frac{\Gamma(a+cj+\lambda_{n-j})\, \Gamma(b+cj)}{\Gamma(a+b+c(n-1+j)+\lambda_{n-j})} \prod_{1 \leqslant i < j \leqslant n} (\lambda_i - \lambda_j + c(j-i))_c. \tag{1.1}$$

Here

$$n \geqslant 1, \qquad c \geqslant 0, \qquad Re(a) > 0, \qquad Re(b) > 0, \tag{1.2}$$

$$\lambda = (\lambda_1, \lambda_2, ..., \lambda_n) \qquad \text{for integers} \quad \lambda_1 \geqslant \cdots \geqslant \lambda_n \geqslant 0, \tag{1.3}$$

$(x)_c = \Gamma(x+c)/\Gamma(x)$, and $s_\lambda^c(t_1, ..., t_n)$ is a normalized Jack polynomial [15A, 15, Eqs. (1.7), (6.28)]. If $\lambda_1 = 0$, then $s_\lambda^c(t_1, ..., t_n) \equiv 1$ and (1.1) reduces to Selberg's integral formula. If $\lambda_1 = 1$, then $s_\lambda^c(t_1, ..., t_n)$ is an elementary symmetric function and (1.1) reduces to Aomoto's extension [3] of Selberg's integral formula. (For q-integral extensions of Selberg's integral formula, see also [7].)

0022-314X/95 $12.00

The primary purpose of this paper is to formulate and prove character sum analogs of (1.1) and of limiting cases of (1.1) for $n=2$; see Theorems 2.1, 5.1, and 5.2. The existence of character sum analogs for $n=2$ raises hope that they may exist for general n, although no analogs have yet been proposed for $n>2$. On the other hand, character sum analogs of Selberg's integral formula and limiting cases were conjectured for general n in 1981 [5, Eqs. (29), (29a), (29b)], and these have finally been proved in [6], due to the remarkable work of Anderson [1]. Modifications of the method in [1] have also led to evaluations of the nongeneric character sum analogs of Selberg's integral [20] that were conjectured in [8].

We now outline our approach for $n=2$. We may assume $\lambda_n=0$ (see [15, Eq. (1.4)]). With this reduction, the case $n=2$ of (1.1) is equivalent to [14, Eq. (1.7)]

$$\int_0^1\int_0^1 (t_1t_2)^{a-1}((1-t_1)(1-t_2))^{b-1}(t_1-t_2)^{2c}\,{}_2F_1\left(\begin{matrix}-\lambda & c\\ 1-\lambda-c\end{matrix}\,\middle|\,\frac{t_1}{t_2}\right)t_2^{\lambda}\,dt_1\,dt_2$$
$$=\frac{2\Gamma(2c+\lambda)\,\Gamma(a+c+\lambda)\,\Gamma(b+c)\,\Gamma(a)\,\Gamma(b)}{\Gamma(c+\lambda)\,\Gamma(a+b+2c+\lambda)\,\Gamma(a+b+c)},\tag{1.4}$$

where we write λ in place of λ_1 for brevity (abusing notation). In order to formulate a character sum analog of (1.4), we first change the variables of integration to the elementary symmetric functions in the t_i (just as we did to formulate an analog of Selberg's integral formula in [5, Eq. (29)]). With the change of variables

$$r=t_1+t_2,\qquad s=t_1t_2,\tag{1.5}$$

(1.4) becomes

$$\int_{r=0}^{2}\int_{s=0}^{r^2/4} s^{a-1}(1-r+s)^{b-1}(r^2-4s)^{c+(\lambda-1)/2}$$
$$\times\,{}_2F_1\left(\begin{matrix}-\lambda/2 & -c-(\lambda-1)/2\\ & -c-\lambda+1\end{matrix}\,\middle|\,\frac{-4s}{r^2-4s}\right)ds\,dr$$
$$=\frac{2\Gamma(2c+\lambda)\,\Gamma(a+c+\lambda)\,\Gamma(b+c)\,\Gamma(a)\,\Gamma(b)}{\Gamma(c+\lambda)\,\Gamma(a+b+2c+\lambda)\,\Gamma(a+b+c)},\tag{1.6}$$

where we have used the quadratic transformation formula [16, p. 252, Eq. (9.6.9)]

$${}_2F_1\left(\begin{matrix}\alpha & \beta\\ \alpha-\beta+1\end{matrix}\,\middle|\,z\right)=(1-z)^{-\alpha}\,{}_2F_1\left(\begin{matrix}\alpha/2 & -\beta+(\alpha+1)/2\\ & \alpha-\beta+1\end{matrix}\,\middle|\,\frac{-4z}{(1-z)^2}\right)\tag{1.7}$$

with $\alpha = -\lambda$, $\beta = c$, $z = t_1/t_2$. Now (1.6) directly suggests a "Selberg–Jack" character sum formula of the form

$$\sum_F A(F(0))\, B(F(1))\, CL\phi(D_F)\, {}_2F_1\left(\begin{matrix}\bar{L} & \bar{C}\bar{L}\phi \\ & \bar{C}\bar{L}^2\end{matrix}\,\middle|\, \frac{-4F(0)}{D_F}\right)$$
$$= \frac{G(C^2L^2)\, G(ACL^2)\, G(BC)\, G(A)\, G(B)}{G(CL^2)\, G(ABC^2L^2)\, G(ABC)}, \tag{1.8}$$

where the sum is over all monic polynomials $F(x) = x^2 - rx + s$ over a finite field $GF(q)$ of characteristic >2, ϕ, A, B, C, $\bar{C}$, L, $\bar{L}$ are characters on $GF(q)$ (corresponding to $1/2$, a, b, c, $-c$, $\lambda/2$, $-\lambda/2$, resp.), $G(A)$ is a Gauss sum (corresponding to $\Gamma(a)$), D_F denotes the discriminant of F, and the ${}_2F_1$ is a suitably normalized hypergeometric character sum [9, 10] (corresponding to the classical ${}_2F_1$ hypergeometric series). This will be made precise in Section 2 (see Theorem 2.1), where it is shown that if the ${}_2F_1$ in (1.8) is defined as in (2.6), then (1.8) holds generically. Theorem 2.1 is proved in Section 4. Related results ("limiting cases") are presented in Section 5.

2. Generic Selberg–Jack Character Sums

Let $GF(q)$ denote the field of q elements, where q is a power of an odd prime p. Let 1 and ϕ denote the trivial and quadratic characters on $GF(q)$, respectively. Throughout, A, B, C, and L will denote characters on $GF(q)$. By convention, $A(0) = 0$, even for $A = 1$. Define $\bar{A}$ by $A\bar{A} = 1$. Define the Gauss sum $G(A)$ and Jacobi sum $J(A, B)$ over $GF(q)$ by

$$G(A) = \sum_m A(m)\, \zeta^{T(m)}, \qquad J(A, B) = \sum_m A(m)\, B(1-m), \tag{2.1}$$

where each sum is over all $m \in GF(q)$, $\zeta = \exp(2\pi i/p)$, and T is the trace map from $GF(q)$ to $GF(p)$. Thus [13, p. 93]

$$J(A, B) = \begin{cases} G(A)\, G(B)/G(AB), & \text{if } AB \neq 1 \\ -A(-1), & \text{if } AB = 1. \end{cases} \tag{2.2}$$

The n-dimensional Selberg character sum $L_n(A, B, C\phi)$ over $GF(q)$ is

$$L_n(A, B, C\phi) := \sum_{\substack{F \\ \deg F = n}} A((-1)^n F(0))\, B(F(1))\, C\phi(D_F), \tag{2.3}$$

where the sum is over all monic polynomials F over $GF(q)$ of degree n and where D_F denotes the discriminant of F. A generic character sum analog of Selberg's integral formula for dimension 2 is [8, Eq. (1.11)]

$$L_2(A, B, C\phi) = \frac{G(C^2)\, G(AC)\, G(BC)\, G(A)\, G(B)}{G(C)\, G(ABC^2)\, G(ABC)} \tag{2.4}$$

when

$$ABC \neq 1, \qquad ABC^2 \neq 1. \tag{2.5}$$

For a proof, see [8, Section 3]. Theorem 2.1 will extend (2.4).

For $0 \neq t \in GF(q)$, define the character sum $P(L, C; t)$ by

$$P(L, C; t) = \frac{qL(-1)\, G(CL\phi)\, CL(t)\{\phi(-t) - \phi(1-t)\}}{G(CL^2)\, G(\bar{L})\, G(\phi)} + \frac{1}{q-1} \frac{G(CL\phi)}{G(CL^2)\, G(\bar{L})} \sum_M \frac{M(-t)\, G(CL^2\bar{M})\, G(\bar{L}M)\, G(\bar{M})}{G(CL\phi\bar{M})}, \tag{2.6}$$

where the sum is over all characters M on $GF(q)$. The sum $P(L, C; t)$ will serve as our desired normalized hypergeometric character sum ${}_2F_1\left(\begin{smallmatrix} L & \bar{C}\bar{L}\phi \\ & \bar{C}\bar{L}^2 \end{smallmatrix} \,\middle|\, t\right)$ in (1.8).

Let T_1 and T_2 denote the first and second terms, respectively, on the right of (2.6). Note that the form of T_2 directly matches that of the classical ${}_2F_1$ hypergeometric series [16, p. 238, Eq. (9.1.1)] (the Gauss sums in T_2 match the gamma functions in the hypergeometric series). The motivation for tacking on the extra term T_1 in (2.6) is as follows. When $\lambda = 0$, the Selberg–Jack integral formula (1.4) reduces to the Selberg integral formula, because the classical ${}_2F_1$ in (1.4) equals 1 for $\lambda = 0$. Choosing $\lambda = 0$ in this ${}_2F_1$ corresponds in the finite field analog to choosing L to be the trivial character in T_2. However, in contrast with the classical analog, T_2 does not equal 1 when L is trivial. Tacking on the term T_1 in (2.6) provides the desired normalization

$$P(1, C; t) = 1, \tag{2.7}$$

so that (2.9) reduces to the Selberg character sum formula (2.4) when $L = 1$. We prove (2.7) in Lemma 3.2.

Theorem 2.1 below gives a precise version of (1.8). It will be proved in Section 4. In some cases, (2.9) holds even when the restriction $(ACL)^2 \neq 1$ in (2.8) is dropped. This happens, e.g., if $L = 1$ or $L = \bar{C}$; the proof follows easily from (2.4)–(2.5) and Lemma 3.2.

THEOREM 2.1. *If*

$$(ACL)^2,\ ABCL,\ ABC^2L^2, \text{ and } ABC \qquad \text{are nontrivial,} \tag{2.8}$$

then

$$\sum_{\substack{F \\ \deg F=2}} A(F(0))\, B(F(1))\, CL\phi(D_F)\, P(L, C; -4F(0)/D_F)$$
$$= \frac{G(C^2L^2)\, G(ACL^2)\, G(BC)\, G(A)\, G(B)}{G(CL^2)\, G(ABC^2L^2)\, G(ABC)}, \tag{2.9}$$

where F *runs through all monic quadratic polynomials over* $GF(q)$ *with* $D_F \neq 0$.

3. LEMMAS

Lemma 3.1 below is the analog of the Gauss duplication formula. It is a special case of the Hasse–Davenport product formula [17, p. 211].

LEMMA 3.1. *For all characters* M,

$$G(M^2)\, G(\phi) = M(4)\, G(M)\, G(M\phi).$$

The following lemma proves (2.7).

LEMMA 3.2. *For all* C *and* $0 \neq t \in GF(q)$,

$$P(L, C; t) = \begin{cases} 1, & \text{if } L = 1 \\ \dfrac{-L(-t)\, G(\phi)}{G(\phi\bar{L})\, G(L)}, & \text{if } L = \bar{C}. \end{cases} \tag{3.1}$$

Proof. For all C, L and all $t \in GF(q)$,

$$P(L, C; t) = \frac{CL^2(-t/4)\, G(C^2L^2)\, G(\bar{C}\bar{L}^2)}{G(\bar{L}^2)\, G(CL^2)}\, P(\bar{C}\bar{L}, C; t). \tag{3.2}$$

This follows easily upon replacing M by MCL^2 in (2.6) and then employing Lemma 3.1. By (3.2) and Lemma 3.1, it suffices to prove (3.1) in the case $L = 1$. By (2.6),

$$P(1, C; t)+\frac{qG(C\phi)\,C(t)\{\phi(-t)-\phi(1-t)\}}{G(C)\,G(\phi)}$$

$$=\frac{-1}{q-1}\frac{G(C\phi)}{G(C)}\sum_M \frac{M(t)\,G(C\bar{M})\,G(\bar{M})\,G(M)\,M(-1)}{G(C\phi\bar{M})}$$

$$=1-\frac{q}{q-1}\frac{G(C\phi)}{G(C)}\sum_M M(t)\frac{G(C\bar{M})}{G(C\phi\bar{M})}$$

$$=1-\frac{q}{q-1}\frac{G(C\phi)\,C(t)}{G(C)}\sum_M M(t)\frac{G(\bar{M})}{G(\phi\bar{M})}$$

$$=1-\frac{q}{q-1}\frac{G(C\phi)\,C(t)}{G(C)\,G(\phi)}\left\{\sum_M M(t)\,J(\phi, \bar{M})-(q-1)\,\phi(-t)\right\},$$

by (2.2) with $A=\phi$, $B=\bar{M}$. Since

$$\sum_M M(t)\,J(\phi, \bar{M})=\sum_m \phi(1-m)\sum_M M(m/t)=(q-1)\,\phi(1-t), \tag{3.3}$$

the result follows.

Lemmas 3.3 and 3.4 below are analogs of classical summation theorems of Gauss and Saalschütz, respectively [19]. Proofs of Lemma 3.3 may be found in [10, 12]. Lemma 3.4 is proved in [11].

LEMMA 3.3. *For all characters A, B, C, D,*

$$\frac{1}{1-q}\sum_M G(AM)\,G(B\bar{M})\,G(CM)\,G(D\bar{M})$$
$$=\frac{G(AB)\,G(AD)\,G(BC)\,G(CD)}{G(ABCD)}+q(q-1)\,AC(-1)\,\delta(ABCD),$$

where δ is defined by

$$\delta(M)=\begin{cases}1, & \text{if } M=1\\ 0, & \text{otherwise.}\end{cases} \tag{3.4}$$

LEMMA 3.4. *For characters A, B, C, D, E, F with $ABCDEF=1$,*

$$\frac{1}{q-1}\sum_M M(-1)\,G(AM)\,G(BM)\,G(CM)\,G(D\bar{M})\,G(E\bar{M})\,G(F\bar{M})$$
$$=-q^2DEF(-1)+q^{-2}DEF(-1)\,G(AD)\,G(BD)\,G(CD)\,G(AE)$$
$$\times\,G(BE)\,G(CE)\,G(AF)\,G(BF)\,G(CF),$$

provided that

$$\{\bar{D}, \bar{E}, \bar{F}\} \neq \{A, B, C\}. \tag{3.5}$$

Lemma 3.5 below extends (2.4). It is the special case of [8, Eq. (2.6)] where $n=2$. For a proof, see [8, p. 116 and Section 3].

LEMMA 3.5. *If*

$$\{A, B, ABC\} \quad \text{is not contained in} \quad \{1, \bar{C}\}, \tag{3.6}$$

then

$$L_2(A, B, C\phi) = \frac{C(-1)\, G(C^2)\, G(AC)\, G(BC)\, G(A)\, G(B)\, G(\bar{A}\bar{B}\bar{C})\, G(\bar{A}\bar{B}\bar{C}^2)}{q^2 G(C)}. \tag{3.7}$$

4. PROOF OF THEOREM 2.1

By (2.6), the left side of (2.9) equals $R+S$, where

$$R = \frac{qL(4)\, G(CL\phi)\, C(-4)}{G(CL^2)\, G(\bar{L})\, G(\phi)} \times \sum_{\deg F=2} ACL(F(0))\, B(F(1))\{\phi(F(0)) - \phi(D_F + 4F(0))\} \tag{4.1}$$

and

$$S = \frac{1}{q-1} \frac{G(CL\phi)}{G(CL^2)\, G(\bar{L})} \times \sum_M \frac{G(CL^2\bar{M})\, G(\bar{M})\, G(\bar{L}M)\, M(4)\, L_2(AM, B, CL\phi\bar{M})}{G(CL\phi\bar{M})} \tag{4.2}$$

with L_2 defined by (2.3). Since

$$\phi(D_F + 4F(0)) = \begin{cases} 0, & \text{if } F(x) = x^2 + m \text{ for some } m \in GF(q), \\ 1, & \text{otherwise,} \end{cases} \tag{4.3}$$

$$R = \frac{qL(4)\, G(CL\phi)\, C(-4)}{G(CL^2)\, G(\bar{L})\, G(\phi)} \left\{ L_2(ACL\phi, B, 1) - L_2(ACL, B, 1) + \sum_m ACL(m)\, B(1+m) \right\}. \tag{4.4}$$

Since $ABCL \neq 1$, the sum on m in (4.4) equals

$$ACL(-1)\,G(ACL)\,G(B)/G(ABCL),$$

by (2.2). By [8, Eq. (1.7)],

$$L_2(ACL\phi, B, 1) = L_2(ACL, B, 1), \tag{4.5}$$

since $(ACL)^2 \neq 1$. Thus (4.4) becomes

$$R = \frac{qCL(4)\,AL(-1)\,G(CL\phi)\,G(ACL)\,G(B)}{G(CL^2)\,G(\bar{L})\,G(\phi)\,G(ABCL)}. \tag{4.6}$$

We proceed to evaluate S. Since $(ACL)^2 \neq 1$ and $ABCL \neq 1$, we see that for all M,

$$\{AM, B, ABCL\} \quad \text{is not contained in} \quad \{1, \bar{C}\bar{L}M\}. \tag{4.7}$$

Thus we can apply Lemma 3.5 to obtain, for all M,

$$L_2(AM, B, CL\bar{M}\phi) = \frac{\begin{pmatrix} CLM(-1)\,G(C^2L^2\bar{M}^2)\,G(ACL)\,G(BCL\bar{M}) \\ \times G(AM)\,G(B)\,G(\bar{A}\bar{B}\bar{C}\bar{L})\,G(\bar{A}\bar{B}\bar{C}^2\bar{L}^2M) \end{pmatrix}}{q^2 G(CL\bar{M})}. \tag{4.8}$$

Substituting (4.8) into (4.2), we obtain

$$\begin{aligned} S = {} & \frac{G(CL\phi)\,CL(-1)\,G(ACL)\,G(B)\,G(\bar{A}\bar{B}\bar{C}\bar{L})}{(q-1)\,q^2 G(CL^2)\,G(\bar{L})} \\ & \times \sum_M \left\{ \frac{M(4)\,G(C^2L^2\bar{M}^2)}{G(CL\bar{M})\,G(CL\bar{M}\phi)} \right\} G(CL^2M)\,G(\bar{M})\,G(\bar{L}M) \\ & \times M(-1)\,G(BCL\bar{M})\,G(AM)\,G(\bar{A}\bar{B}\bar{C}^2\bar{L}^2M). \end{aligned} \tag{4.9}$$

The expression in braces in (4.9) equals $LC(4)/G(\phi)$ by Lemma 3.1, so (4.9) becomes

$$\begin{aligned} S = {} & \frac{G(CL\phi)\,CL(-4)\,G(ACL)\,G(B)\,G(\bar{A}\bar{B}\bar{C}\bar{L})}{(q-1)\,q^2 G(CL^2)\,G(\bar{L})\,G(\phi)} \\ & \times \sum_M M(-1)\,G(AM)\,G(\bar{L}M)\,G(\bar{A}\bar{B}\bar{C}^2\bar{L}^2M) \\ & \times G(\bar{M})\,G(CL^2\bar{M})\,G(BCL\bar{M}). \end{aligned} \tag{4.10}$$

Since $ABC^2L^2 \neq 1$, $ABC \neq 1$, $ACL \neq 1$, we have

$$\{\bar{A}, L, ABC^2L^2\} \neq \{1, CL^2, BCL\}. \tag{4.11}$$

Thus we can apply Lemma 3.4 in (4.10) to obtain

$$\begin{aligned} S = {} & \frac{BL(-1)\, CL(-4)\, G(CL\phi)\, G(ACL)\, G(B)\, G(\bar{A}\bar{B}\bar{C}\bar{L})}{q^2 G(CL^2)\, G(\bar{L})\, G(\phi)} \\ & \times \{-q^2 + q^{-2} G(A)\, G(ACL^2)\, G(ABCL)\, G(\bar{L})\, G(CL)\, G(BC) \\ & \times G(\bar{A}\bar{B}\bar{C}^2\bar{L}^2)\, G(\bar{A}\bar{B}\bar{C})\, G(\bar{A}\bar{C}\bar{L})\}. \end{aligned} \tag{4.12}$$

Combining (4.6) and (4.12), we have, since $ABCL \neq 1$,

$$\begin{aligned} R + S = {} & \frac{BC(-1)\, CL(4)\, G(CL\phi)\, G(ACL)\, G(B)\, G(\bar{A}\bar{B}\bar{C}\bar{L})}{q^4 G(CL^2)\, G(\bar{L})\, G(\phi)} \\ & \times G(A)\, G(ACL^2)\, G(ABCL)\, G(\bar{L})\, G(CL)\, G(BC) \\ & \times G(\bar{A}\bar{B}\bar{C}^2\bar{L}^2)\, G(\bar{A}\bar{B}\bar{C})\, G(\bar{A}\bar{C}\bar{L}). \end{aligned} \tag{4.13}$$

Since $ABC^2L^2 \neq 1$, $ABCL \neq 1$, $ACL \neq 1$, $ABC \neq 1$, we have

$$R + S = \frac{CL(4)\, G(CL\phi)\, G(B)\, G(A)\, (ACL^2)\, G(CL)\, G(BC)}{G(CL^2)\, G(\phi)\, G(ABC^2L^2)\, G(ABC)}. \tag{4.14}$$

By Lemma 3.1, $G(C^2L^2)\, G(\phi) = CL(4)\, G(CL)\, G(CL\phi)$, so the result follows from (4.14).

5. Limiting Cases

A character sum analog of a limiting case of Selberg's n-dimensional integral formula [5, Eq. (29a)] is proved in [6]. For $n = 2$, this analog states that for all characters A, C,

$$\sum_{\substack{F \\ \deg F = 2}} A(F(0))\, C\phi(D_F)\, \zeta^{T(\alpha)} = G(C^2)\, G(A)\, G(AC)/G(C), \tag{5.1}$$

where

$$F(x) = x^2 + \alpha x + \beta, \qquad \alpha, \beta \in GF(q). \tag{5.2}$$

The following theorem generalizes (5.1).

THEOREM 5.1. *For all characters A, C, L,*

$$\sum_{\substack{F \\ \deg F=2}} A(F(0))\, CL\phi(D_F)\, \zeta^{T(\alpha)} P(L, C; -4F(0)/D_F)$$
$$= G(C^2L^2)\, G(A)\, G(ACL^2)/G(CL^2). \tag{5.3}$$

Proof. The proof is similar to that of Theorem 2.1, except that Lemma 3.3 is used instead of Lemma 3.4.

A character sum analog of a recent n-dimensional integral formula of Selberg [4, Eq. (2.1)] is proved in [6]. For $n=2$, this analog states that for all characters A, B, C,

$$\sum_{\substack{F \\ \deg F=2}} A(F(0))\, B(1+\alpha)\, C\phi(D_F)$$
$$\begin{cases} \dfrac{G(\bar{B}\bar{A}^2\bar{C}^2)\, G(C^2)\, G(A)\, G(AC)}{G(\bar{B})\, G(C)}, & \text{if } B \neq 1 \\[2ex] \dfrac{G(B)\, G(C^2)\, G(A)\, G(AC)}{G(BA^2C^2)\, G(C)}, & \text{if } BA^2C^2 \neq 1, \end{cases} \tag{5.4}$$

where α and F are given by (5.2). The following theorem generalizes (5.4).

THEOREM 5.2. *For all characters A, B, C, L,*

$$\sum_{\substack{F \\ \deg F=2}} A(F(0))\, B(1+\alpha)\, CL\phi(D_F)\, P(L, C; -4F(0)/D_F)$$
$$\begin{cases} \dfrac{G(\bar{B}\bar{A}^2\bar{C}^2\bar{L}^2)\, G(C^2L^2)\, G(A)\, G(ACL^2)}{G(\bar{B})\, G(CL^2)}, & \textit{if } B \neq 1 \\[2ex] \dfrac{G(B)\, G(C^2L^2)\, G(A)\, G(ACL^2)}{G(BA^2C^2L^2)\, G(CL^2)}, & \textit{if } BA^2C^2L^2 \neq 1, \end{cases} \tag{5.5}$$

Proof. This follows from Theorem 5.1 in the same way that (5.4) follows from (5.1); see the proof of [4, Theorem 2.2].

ACKNOWLEDGMENTS

It was a time-consuming hit-or-miss process to find an appropriate definition for the ${}_2F_1$ in (1.8) among the endless variety of possible normalizations. We are grateful to H. Stark for his intuition in this regard. We are also indebted to J. Greene for pointing out Lemma 3.4.

REFERENCES

1. G. ANDERSON, The evaluation of Selberg sums, *C. R. Acad. Sci. Paris Sér. I Math.* **311** (1990), 469–472.
2. G. ANDREWS, *q* series: Their development and applications in analysis, number theory, combinatorics, physics, and computer algebra, *CBMS Regional Conf. Ser. in Math.* No. 66 (1986).
3. K. AOMOTO, Jacobi polynomials associated with Selberg's integral, *SIAM J. Math. Anal.* **18** (1987), 545–549.
4. J. AUTUORE AND R. EVANS, Evaluations of Selberg character sums, *in* "Analytic Number Theory" (B. C. Berndt *et al.*, Eds.), pp. 13–21. Birkhäuser, Boston, 1990.
5. R. EVANS, Identities for products of Gauss sums over finite fields, *Enseign. Math.* **27** (1981), 197–209.
6. R. EVANS, The evaluation of Selberg character sums, *Enseign. Math.* **37** (1991), 235–248.
7. R. EVANS, Multidimensional beta and gamma integrals, *Contemporary Math.* **166** (1994), 341–357.
8. R. EVANS AND W. ROOT, Conjectures for Selberg character sums, *J. Ramanujan Math. Soc.* **3**(1) (1988), 111–128.
9. J. GREENE, Hypergeometric functions over finite fields, *Trans. Amer. Math. Soc.* **301** (1987), 77–101.
10. J. GREENE, Hypergeometric functions over finite fields and representations of $SL(2, q)$, *Rocky Mountain J. Math.* **23** (1993), 547–568.
11. J. GREENE, The Bailey transform over finite fields, to appear.
12. A. HELVERSEN-PASOTTO, L'identité de Barnes pour les corps finis, *C. R. Acad. Sci. Paris. Sér. A.* **286** (1978), 297–300.
13. K. IRELAND AND M. ROSEN, A classical introduction to modern number theory, "Graduate Texts in Mathematics," Vol. 84, Springer-Verlag, New York, 1982.
14. K. KADELL, The *q*-Selberg polynomials for $n = 2$, *Trans. Amer. Math. Soc.* **310** (1988), 535–553.
15. K. KADELL, The Selberg–Jack symmetric functions, preprint.

15A. J. KANEKO, Selberg integrals and hypergeometric functions associated with Jack polynomials, *SIAM J. Math. Analysis* **24** (1993), 1086–1110.

16. N. LEBEDEV, "Special Functions and Their Applications," Dover, New York, 1972.
17. R. LIDL AND H. NIEDERREITER, "Finite Fields," Addison–Wesley, New York, 1983.
18. A. SELBERG, Bemerkninger om et multipelt integral, *Norsk. Mat. Tidsskr.* **26** (1944), 71–78.
19. L. SLATER, "Generalized Hypergeometric Functions," Cambridge Univ. Press, Cambridge, 1966.
20. P. VAN WAMELEN, Proof of Evans-Root conjectures for Selberg character sums, *J. London Math. Soc.* **48** (1993), 415–426.