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POLYNOMIALS WITH NONNEGATIVE COEFFICIENTS WHOSE ZEROS HAVE MODULUS ONE*

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Abstract. Define $p(z) = \prod_{j=0}^{n-1} \left(z - e^{i(\theta+\alpha j)}\right)\left(z - e^{-i(\theta+\alpha j)}\right)$ for $\alpha > 0$ and $\theta \geq 0$ with $\pi/2 - (n-1)\alpha/2 \leq \theta \leq \pi - (n-1)\alpha/2$. It is proved that if $0 < \alpha < \pi/n$, then the $2n+1$ coefficients of $p(z)$ are all positive. It is also proved that if for some point θ, all coefficients of $p(z)$ are nonnegative, then each coefficient is an increasing function of θ in a neighborhood of this point. A similar result is conjectured for more general polynomials $p(z)$.

Key words. orthogonal polynomials, q-ultraspherical polynomials, absolutely monotonic polynomials

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1. Introduction.

For

$$\alpha > 0 \quad \text{and} \quad \theta \geq 0, \tag{1.1}$$

consider the monic polynomial $p(z)$ of degree $2n$ whose zeros consist of the n equally spaced points

$$\exp(i(\theta + \alpha j)), \qquad 0 \leq j \leq n-1, \tag{1.2}$$

along with their n complex conjugates, i.e.,

$$p(z) = \prod_{j=0}^{n-1} \left(z - e^{i(\theta+\alpha j)}\right)\left(z - e^{-i(\theta+\alpha j)}\right). \tag{1.3}$$

We assume *throughout* that the variable θ in (1.3) is restricted to the interval

$$\pi/2 - (n-1)\alpha/2 \leq \theta \leq \pi - (n-1)\alpha/2. \tag{1.4}$$

Equivalently,

$$\pi/2 \leq \theta + (n-1)\alpha/2 \leq \pi, \tag{1.5}$$

so that the geometric mean of the n zeros in (1.2) lies in the second quadrant. Condition (1.5) automatically holds, for example, if each of the n zeros in (1.2) has Argument $\in (0, \pi)$ and the coefficient of z in $p(z)$ is positive; this is easily seen from (2.12) and (2.17). When (1.5) holds, the geometric mean of the n zeros in (1.2) is closer to -1 than to $+1$, and it moves (together with at least half of the zeros of $p(z)$) towards -1 along the unit circle as θ increases.

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The coefficients of $p(z)$ are not necessarily increasing functions of θ, even if each of the n zeros in (1.2) has Argument $\in (0, \pi)$ (in which case each of the n quadratic factors in (1.3) has increasing coefficients). For example, if $n = 3$, $\alpha = 5\pi/12$, then the coefficient of z^3 in $p(z)$ is negative and *decreasing* at $\theta = \pi/8$, while $\pi/8$ is in the interval (1.4). However, the following theorem holds for all n. The proof, given in §3, depends on properties of q-ultraspherical polynomials discussed in §2.

THEOREM 1. *If for some nonnegative $\theta = \theta_0$ in the interval* (1.4), *all coefficients of $p(z)$ are nonnegative, then they are each increasing functions of θ for $\theta_0 \leq \theta < \pi - (n-1)\alpha/2$. Except for the coefficients* 1 *of the leading and constant terms, the coefficients are in fact strictly increasing, unless $\alpha = 2\pi/n$.*

For $\alpha = 2\pi/n$, we have

$$p(z) = z^{2n} - 2\cos(\theta n) z^n + 1,$$

which has nonnegative coefficients for $\pi/(2n) \leq \theta \leq \pi/n$, but if $n > 1$, the coefficient of z is zero, which is not strictly increasing. This formula for $p(z)$ is proved in §3 (see (3.10)).

Consider for the moment the general polynomial

$$P(z) = \prod_{j=0}^{n-1} \left(z - e^{i(\theta + a_j)}\right)\left(z - e^{-i(\theta + a_j)}\right) \tag{1.6}$$

where

$$\theta \geq 0, \qquad 0 = a_0 < a_1 < \cdots < a_{n-1}. \tag{1.7}$$

The polynomial $P(z)$ reduces to $p(z)$ when $a_j = j\alpha$, $0 \leq j \leq n-1$. In view of Theorem 1, we might ask if nonnegativity of the coefficients of $P(z)$ for some $\theta = \theta_0$ always implies that the coefficients are increasing for $\theta \geq \theta_0$, when θ is restricted to the interval

$$\pi/2 - (a_1 + \cdots + a_{n-1})/n \leq \theta \leq \pi - (a_1 + \cdots + a_{n-1})/n. \tag{1.8}$$

The answer is no. For example, if $n = 3$, $a_1 = \pi/2$, $a_2 = 7\pi/12$, then the coefficients of $P(z)$ are all positive for $\pi/4 < \theta < 23\pi/36$, yet the coefficients of z^2, z^3, z^4 are each decreasing at $\theta = 2$. However, we believe the following.

CONJECTURE. *If the coefficients of $P(z)$ are all nonnegative for some $\theta = \theta_0 \geq 0$, then they are each increasing functions of θ on the interval $\theta_0 \leq \theta < \pi - a_{n-1}$.*

For convenient application of Theorem 1, we would like to have a simple necessary condition for the nonnegativity of the coefficients of $p(z)$. This is given in Theorem 2.

THEOREM 2. *Suppose that*

$$0 < \alpha < \pi/n. \tag{1.9}$$

Then each coefficient of $p(z)$ is positive (*and hence increasing in θ, by Theorem* 1).

This theorem was motivated by the fact that for sufficiently small α, all zeros of $p(z)$ are closer to -1 than to $+1$ (because of (1.5)), and so all coefficients of $p(z)$ are positive. The question is how small α must be.

For $n > 1$, the upper bound in (1.9) is best possible, i.e., if $\alpha > \pi/n$, the coefficients of $p(z)$ cannot all be positive on the interval (1.4). If $\alpha \geq 2\pi/n$, there is *no* θ

in the interval (1.4) for which all coefficients of $p(z)$ are positive. If $\pi/n \le \alpha < 2\pi/n$, the coefficients of $p(z)$ are all positive only on a subinterval

$$r_\alpha < \theta < \pi - (n-1)\alpha/2 \tag{1.10}$$

of the interval (1.4). These remarks will be proved in §4. Also in §4 we prove Theorem 2 and the following related result.

THEOREM 3. *Let* $0 < \alpha < \pi/n$. *Then all coefficients of*

$$p(u,v) := \prod_{j=(1-n)/2}^{(n-1)/2} (1 + ue^{i\alpha j} + ve^{-i\alpha j}) \tag{1.11}$$

are positive, *i.e.*,

$$p(u,v) = \sum_{\substack{0\le r,s\le n \\ r+s\le n}} a_{rs}u^r v^s, \qquad a_{rs} > 0. \tag{1.12}$$

(The variable j in (1.11) ranges over halves of odd integers if n is even.)

As an application of Theorem 2, we give in §5 a short proof of Theorem 4 below in the special case

$$f(z) = (z^{mk} - 1)\,/\,(z^k - 1)\,, \tag{1.13}$$

where m, k are positive integers.

THEOREM 4. *Let* $f(z)$ *denote a monic polynomial of degree* N *with nonnegative coefficients and with zeros* $z_1, z_2, \cdots, z_N$. *For fixed* $t \ge 0$, *write*

$$f_t(z) = \prod_{\substack{1\le j\le N \\ |\mathrm{Arg}\, z_j|>t}} (z - z_j). \tag{1.14}$$

Then if $f(z) \ne f_t(z)$, *all coefficients of* $f_t(z)$ *are positive.*

Theorem 4 had been open for several years until a proof was found recently by Barnard et al. [2].

In the special cases $f(z) = z^N + 1$ or $f(z) = 1 + z + \cdots + z^N$, we can say a bit more about the polynomials $f_t(z)$ in (1.14), namely, the following theorem [4].

THEOREM 5. *If* $f(z) = z^N + 1$ *or* $f(z) = 1 + z + \cdots + z^N$, *and if* $f_t(z) \ne f(z)$, *then* $f_t(z)$ *is a strictly unimodal polynomial.* (*In particular, all coefficients of* $f_t(z)$ *are* ≥ 1.)

If $f(z)$ is given by (1.13), it is not generally true that $f_t(z)$ is unimodal when $f_t(z) \ne f(z)$.

2. The coefficients of $p(z)$ in terms of q-ultraspherical polynomials. We will use the following additional notation throughout:

$$q = e^{i\alpha}, \tag{2.1}$$

$$\beta = \theta + (n-1)\alpha/2 - \pi/2, \tag{2.2}$$

and

$$x = \sin \beta. \tag{2.3}$$

Observe that (1.5) is equivalent to

$$0 \leq \beta \leq \pi/2, \tag{2.4}$$

which implies that

$$0 \leq x \leq 1, \qquad \frac{dx}{d\theta} \geq 0. \tag{2.5}$$

In order to relate $p(z)$ to q-ultraspherical polynomials (see (2.12)–(2.13)), we begin by replacing j by $j + (n-1)/2$ in (1.3) to obtain

$$p(z) = \prod_{j=(1-n)/2}^{(n-1)/2} \left(z - e^{i(\beta+\alpha j+\pi/2)}\right)\left(z - e^{-i(\beta+\alpha j+\pi/2)}\right). \tag{2.6}$$

Since the range of values of j in (2.6) is symmetric about zero, we have

$$\begin{aligned} p(z) &= \prod_{j=(1-n)/2}^{(n-1)/2} \left(z - e^{i(\beta+\alpha j+\pi/2)}\right)\left(z - e^{-i(\beta-\alpha j+\pi/2)}\right) \\ &= \prod_{j=(1-n)/2}^{(n-1)/2} \left(z^2 - 2zq^j \cos(\beta+\pi/2) + q^{2j}\right) \\ &= \prod_{j=(1-n)/2}^{(n-1)/2} \left(z^2 + 2zq^j \sin \beta + q^{2j}\right). \end{aligned} \tag{2.7}$$

Replace j by $-j$ and multiply each factor by q^{2j} to obtain

$$p(z) = \prod_{j=(1-n)/2}^{(n-1)/2} \left(z^2 q^{2j} + 2zxq^j + 1\right). \tag{2.8}$$

Note that the coefficients of $p(z)$ are symmetric about the middle, as

$$z^{2n} p(1/z) = p(z), \tag{2.9}$$

and the leading and constant coefficients of $p(z)$ are 1 for all θ, α.

The generating function for the q-ultraspherical polynomials $C_k(x;t|q)$ is [1, eq. (3.4), p. 179]

$$\sum_{k=0}^{\infty} C_k(x;t|q) w^k = \prod_{k=0}^{\infty} \frac{(1 - 2twxq^k + t^2w^2q^{2k})}{(1 - 2wxq^k + w^2q^{2k})}, \qquad 0 < q < 1. \tag{2.10}$$

In particular, with $t = q^{-n}$,

$$\sum_{k=0}^{\infty} C_k(x; q^{-n}|q) w^k = \prod_{k=-n}^{-1} (1 - 2wxq^k + w^2 q^{2k}) . \tag{2.11}$$

The polynomials $C_k(x; q^{-n}|q)$ are well defined by (2.11) for $q = e^{i\alpha}$. Replace w by $-zq^{(n+1)/2}$ in (2.11) and use (2.8) to see that

$$p(z) = \sum_{k=0}^{2n} E_k\left(x; q^{-n}|q\right) z^k, \tag{2.12}$$

where

$$E_k := E_k(x) = E_k\left(x; q^{-n}|q\right) = (-1)^k q^{k(n+1)/2} C_k\left(x; q^{-n}|q\right) . \tag{2.13}$$

The $C_k(x; t|q)$ satisfy the recurrence relation [1, eq. (1.1), p. 176]

$$2x(1 - tq^k) C_k(x; t|q) = \left(1 - q^{k+1}\right) C_{k+1}(x; t|q) + \left(1 - t^2 q^{k-1}\right) C_{k-1}(x; t|q) \tag{2.14}$$

for $k \geq 1$, with

$$C_0(x; t|q) = 1, \qquad C_1(x; t|q) = 2x(1 - t)/(1 - q). \tag{2.15}$$

In view of (2.1) and (2.13)–(2.15), the E_k satisfy the recurrence

$$\begin{aligned} E_k &= 2x \frac{\sin((n+1-k)\alpha/2)}{\sin(k\alpha/2)} E_{k-1} \\ &\quad + \frac{\sin((2n+2-k)\alpha/2)}{\sin(k\alpha/2)} E_{k-2} \qquad (k \geq 2) \end{aligned} \tag{2.16}$$

with

$$E_0 = 1, \qquad E_1 = 2x \frac{\sin(n\alpha/2)}{\sin(\alpha/2)} . \tag{2.17}$$

3. Proof of Theorem 1. Theorem 1 is trivial for $n = 1$, so let $n > 1$. For brevity, write

$$A_k = \frac{\sin\left((n+1-k)\alpha/2\right)}{\sin(k\alpha/2)}, \quad B_k = \frac{\sin\left((2n+2-k)\alpha/2\right)}{\sin(k\alpha/2)}, \quad k \geq 1, \tag{3.1}$$

so by (2.16),

$$E_k = 2x A_k E_{k-1} + B_k E_{k-2}, \qquad k \geq 2. \tag{3.2}$$

By hypothesis, for some x_0 with $0 \leq x_0 < 1$,

$$E_k(x_0) \geq 0 \quad \text{for } 0 \leq k \leq 2n. \tag{3.3}$$

By (2.9), it suffices to show that the polynomials $E_k(x)$ are strictly increasing on $x_0 < x < 1$ for $1 \leq k \leq n$.

Case 1. $\alpha < 2\pi/n$. In this case,

$$A_k > 0 \quad \text{for } 1 \le k \le n. \tag{3.4}$$

In particular, the leading coefficient of $E_k(x)$ is positive for each k, $1 \le k \le n$.

Suppose there is an integer m with $2 \le m \le n$ such that

$$B_m < 0, \tag{3.5}$$

and choose the maximal such m. By (3.1),

$$B_k < 0 \quad \text{for } 2 \le k \le m. \tag{3.6}$$

By (3.2) and Favard's theorem [3, Thm. 4.4, p. 21], $E_1, E_2, \cdots, E_m$ are orthogonal polynomials with respect to a positive-definite operator. Thus we can apply the theorem on separation of zeros [3, Thm. 5.3, p. 28] to conclude that the zeros of $E_1, \cdots, E_m$ are all real and simple, and that a zero of E_{k-1} lies strictly between every two consecutive zeros of E_k, $2 \le k \le m$.

We proceed to prove by induction on k that if $1 \le k \le m$, then the largest zero of E_k is $\le x_0$. This holds for $k = 1$ since $E_1 = 2A_1x$ and $0 \le x_0$. Let $k > 1$. By induction hypothesis, the largest zero of E_{k-1} is $\le x_0$, so by separation of zeros, x_0 exceeds the second largest zero of E_k. For x between the largest and second largest zeros of E_k, $E_k(x)$ is negative. Thus, by (3.3), the largest zero of E_k is $\le x_0$, and the induction is complete.

It follows for $1 \le k \le m$ that

$$E_k(x) = c_k \prod_{j=1}^{k} (x - \alpha_{jk}) \tag{3.7}$$

with $c_k > 0$ and $\alpha_{jk} \le x_0$ $(1 \le j \le k)$. Thus $E_k(x)$ is strictly increasing on $x_0 < x < 1$ for $1 \le k \le m$.

If there is no integer m with $2 \le m \le n$ for which (3.5) holds, set $m = 1$. It remains to prove that $E_k(x)$ is strictly increasing on $x_0 < x < 1$ for $n \ge k > m$. This follows from (3.2), since $A_k > 0$ and $B_k \ge 0$.

Case 2. $\alpha = 2\pi/n$. In this case, by (2.16) and (2.17), $E_1(x) = E_2(x) = \cdots = E_{n-1}(x) = 0$. Thus by (2.9) and (2.12),

$$p(z) = z^{2n} + E_n z^n + 1. \tag{3.8}$$

It is easily seen from (1.3) that

$$p(1) = (e^{i\theta n} - 1)(e^{-i\theta n} - 1) = 2 - 2\ \cos(\theta n). \tag{3.9}$$

By (3.8) and (3.9), $E_n = -2\cos(\theta n)$, so

$$p(z) = z^{2n} - 2\cos(\theta n) z^n + 1. \tag{3.10}$$

For $\pi/(2n) \le \theta \le \pi/n$, the coefficients of $p(z)$ are nonnegative and they are increasing functions of θ.

Case 3. $\alpha > 2\pi/n$. In this case, $x_0 > 0$ by (2.2) and (2.3). Moreover, by (1.1) and (1.5), we may suppose that

$$2\pi/n < \alpha < 2\pi/(n-1). \tag{3.11}$$

By (2.17) and (3.11),

$$E_1(x_0) = 2x_0 \ \sin(n\alpha/2)/\sin(\alpha/2) < 0. \tag{3.12}$$

This contradicts (3.3), so Case 3 is vacuous.

4. Proofs of Theorems 2 and 3.

Proof of Theorem 2. Let $0 < \alpha < \pi/n$. By (2.9) and (2.12), it suffices to prove

$$E_k > 0, \qquad 0 \le k \le n. \tag{4.1}$$

This follows for $k = 0, 1$ by (2.17). For $2 \le k \le n$, all sines in (3.1) are positive, so

$$A_k > 0, \quad B_k > 0 \quad \text{for } 2 \le k \le n. \tag{4.2}$$

Thus (4.1) follows by (3.2) and induction on k.

Proof of Theorem 3. Let $0 < \alpha < \pi/n$. The proof of (4.1) actually yields the stronger result

$$E_k = \sum_{i=0}^{M} b_{ik} x^i, \qquad 0 \le k \le 2n, \tag{4.3}$$

with

$$\begin{aligned} b_{ik} &> 0, \quad \text{if } i \equiv k (\text{mod } 2), \\ b_{ik} &= 0, \quad \text{otherwise}, \end{aligned} \tag{4.4}$$

where

$$M = \min(k,\ 2n-k). \tag{4.5}$$

Thus, by (2.8) and (2.12),

$$p(z) = \prod_{j=(1-n)/2}^{(n-1)/2} (z^2 q^j + 2zx + q^{-j}) = \sum_{k=0}^{2n} \sum_{i=0}^{M} b_{ik} x^i z^k. \tag{4.6}$$

Replace x by $x/(2z)$ to get

$$\sum_{k=0}^{2n} \sum_{i=0}^{M} b_{ik} 2^{-i} x^i z^{k-i} = \prod_{j=(1-n)/2}^{(n-1)/2} (z^2 q^j + x + q^{-j}) \,. \tag{4.7}$$

Replace z^2 by z, then x by x^{-1}, and multiply by x^n to get

$$\sum_{k=0}^{2n} \sum_{i=0}^{M} b_{ik} 2^{-i} x^{n-i} z^{(k-i)/2} = \prod_{j=(1-n)/2}^{(n-1)/2} (zxq^j + 1 + xq^{-j}) \,. \tag{4.8}$$

Replace z by z/x to get

$$\sum_{k=0}^{2n}\sum_{i=0}^{M} b_{ik}2^{-i}x^{n-(i+k)/2}z^{(k-i)/2} = \prod_{j=(1-n)/2}^{(n-1)/2} (zq^j + 1 + xq^{-j})\,. \tag{4.9}$$

Now (1.12) follows easily from (4.9), completing the proof of Theorem 3.

We close this section by proving the remarks made in §1 between the statements of Theorems 2 and 3.

Let $n > 1$. Then the upper bound π/n in (1.9) is best possible. For, if α is slightly larger than π/n, then $E_2 < 0$ for sufficiently small x, since

$$E_2 = 4x^2\frac{\sin(n\alpha/2)\sin((n-1)\alpha/2)}{\sin(\alpha/2)\sin(\alpha)} + \frac{\sin(n\alpha)}{\sin(\alpha)}. \tag{4.10}$$

If $\alpha \geq 2\pi/n$, there is no θ in the interval (1.4) for which all coefficients of $p(z)$ are positive, by (3.11) and (3.12). Finally, suppose that

$$0 < \alpha < 2\pi/n. \tag{4.11}$$

Then all coefficients of $p(z)$ are positive on a small interval (1.10), i.e., for x sufficiently close to 1. To see this, it suffices to show that when $x = 1$ (and (4.11) holds), all coefficients of $p(z)$ are positive.

By (2.8), when $x = 1$,

$$p(z) = \prod_{j=(1-n)/2}^{(n-1)/2} (q^j z + 1)^2, \tag{4.12}$$

so

$$p(z) = \left(\sum_{\nu=0}^{n} C(n,\nu)z^\nu\right)^2, \tag{4.13}$$

where the $C(n,\nu)$ are central Gaussian coefficients (see [5, p. 449]). By (4.11) and Theorem 3 of [5, p. 449], all of the $C(n,\nu)$ are positive. Thus, by (4.13), all coefficients of $p(z)$ are positive when $x = 1$, $0 < \alpha < 2\pi/n$.

5. Application to Theorem 4. Let $f(z), f_t(z)$ be given by (1.13) and (1.14), and suppose that $f(z) \neq f_t(z)$. We will use Theorem 2 to show that all coefficients of $f_t(z)$ are positive.

Case 1. $t < 2\pi/k$. We have

$$f(z) = g(z)/h(z), \tag{5.1}$$

where

$$g(z) = \frac{z^{mk}-1}{z-1}, \qquad h(z) = \frac{1-z^k}{1-z}, \tag{5.2}$$

so

$$f_t(z) = g_t(z)/h_t(z). \tag{5.3}$$

However, in Case 1, $h_t(z) = h(z)$, so by (5.3),

$$f_t(z) = g_t(z)/h(z) = (g_t(z)(1-z))\left(1 + z^k + z^{2k} + \cdots\right). \tag{5.4}$$

Let

$$d = \text{degree}\,(g_t(z)). \tag{5.5}$$

By Theorem 5 with $N = mk$, $g_t(z)$ is strictly unimodal, so all terms of $g_t(z)(1-z)$ of degree $\le d/2$ have positive coefficients. Therefore, by (5.4), all terms of $f_t(z)$ of degree $\le d/2$ have positive coefficients. However, $f_t(z)$ has degree $d-(k-1) \le d$ by (5.4), so since the coefficients of $f_t(z)$ are symmetric about the middle one, they are all positive.

Case 2. $t \ge 2\pi/k$. If m is even, say $m = 2M$, then

$$f(z) = \frac{z^{Mk}-1}{z^k-1} \cdot (z^{Mk}+1). \tag{5.6}$$

Applying Theorem 5, we could then deduce the result by induction on m. Thus assume that m is odd, so -1 is not a zero of $f(z)$. We have

$$f(z) = \prod_{r=1}^{m-1} A^{(r)}(z), \tag{5.7}$$

where

$$A^{(r)}(z) = \prod_{\substack{0<\nu<mk/2 \\ \nu\equiv r \pmod m}} \left(z - e^{2\pi i\nu/mk}\right)\left(z - e^{-2\pi i\nu/mk}\right). \tag{5.8}$$

Thus,

$$f_t(z) = \prod_{r=1}^{m-1} A_t^{(r)}(z), \tag{5.9}$$

with

$$A_t^{(r)}(z) = \prod_{\substack{mkt/2\pi<\nu<mk/2 \\ \nu\equiv r \pmod m}} \left(z - e^{2\pi i\nu/mk}\right)\left(z - e^{-2\pi i\nu/mk}\right). \tag{5.10}$$

For any fixed r, the zeros of $A_t^{(r)}(z)$ on the upper half of the unit circle can be written in the form

$$\exp(i(\theta_r + \alpha j)), \qquad 0 \le j \le n_r - 1, \tag{5.11}$$

where

$$\theta_r > t \ge 2\pi/k = \alpha \tag{5.12}$$

and

$$\theta_r + \alpha(n_r - 1) < \pi < \theta_r + \alpha n_r. \tag{5.13}$$

Therefore $A_t^{(r)}(z)$ has the same form as $p(z)$ in (1.3), and furthermore,

$$\pi/2 < \theta_r + (n_r - 1)\alpha/2 < \pi \tag{5.14}$$

as in (1.5). Since, moreover, $0 < \alpha < \pi/n_r$, Theorem 2 implies that all coefficients of $A_t^{(r)}(z)$ are positive. Thus all coefficients of $f_t(z)$ are positive by (5.9).

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