JOURNAL OF RESEARCH of the National Bureau of Standards
Vol. 84, No. 2, March–April 1979

Non-Free Groups Generated By Two Parabolic Matrices*

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November 11, 1978

In 1974, M. Newman conjectured that for any root of unity ζ, the matrix group generated by $\begin{pmatrix} 1 & \zeta \\ 0 & 1 \end{pmatrix}$ and $\begin{pmatrix} 1 & 0 \\ \zeta & 1 \end{pmatrix}$ is non-free. This conjecture is proved here.

Key words: Free groups, matrix groups, roots of unity.

Given a complex number ζ, let G be the group generated by the two matrices

$$A = \begin{pmatrix} 1 & \zeta \\ 0 & 1 \end{pmatrix}, B = \begin{pmatrix} 1 & 0 \\ \zeta & 1 \end{pmatrix}.$$

The problem of characterizing those values of ζ for which G is free has been extensively studied; see [1], [2], [3], and the references therein.

From now on, let ζ denote a primitive q-th root of 1. Newman [4][1] conjectured that G is non-free for all q. The purpose of this note is to prove that conjecture. Our method is similar to that in [4]. In contrast with our result, Brenner and Charnow [2, Theorem 6.1] have proved that the *semigroup* generated by A and B is free if $q \triangle \{3, 4, 6\}$.

THEOREM: G is non-free for every primitive q-th root of unity ζ.

PROOF: *For* $m \geq 1$, *inductively define* $K_m \in G$ *as follows:*

$$K_1 = B, K_{m+1} = K_m A^{-1} K_m^{-1}.$$

Write

$$K_n = \begin{pmatrix} a_n & b_n \\ c_n & d_n \end{pmatrix}.$$

As noted in [4], it is easily seen that K_n has trace 2 and determinant 1, and that K_n is determined by the equalities

$$a_n = \zeta^{2^n} \sum_{k=1}^{n} \zeta^{-2^k}$$

and

$$c_n = \zeta^{2^n - 1}.$$

For $n, m \geq 1$, define $K(n, m) \triangle G$ inductively as follows:

$$K(n, 0) = K_n, K(n, m) = K(n, m-1) A K^{-1}(n, m-1).$$

As a formal word in A and B, no cancellation occurs in $K(n, m)$, and $K(n, m)$ has length $2^{n+m} - 1$.

AMS Subject Classification: 20F05; 20H10

* Invited paper

[1] Figures in brackets indicate literature references at the end of this paper.

For $n, m \geq 1$, write

$$K(n, m) = \begin{pmatrix} a(n, m) & b(n, m) \\ c(n, m) & d(n, m) \end{pmatrix}.$$

Observe that $K(n, m)$ has trace 2 and determinant 1, and $K(n, m)$ is determined by the equalities

$$a(n, m) = -\zeta^{2^{n+m}} \left(\sum_{k=1}^{n} \zeta^{-2^k} - \sum_{k=n+1}^{n+m} \zeta^{-2^k} \right)$$

and

$$c(n, m) = -\zeta^{2^{n+m}-1}.$$

Write $q = 2^j u$ with u odd. Let t denote the order of 2 (mod u), so that $2^{j+t} \equiv 2^j \pmod{q}$. We have

$$c(j + t, t + 1) = -\zeta^{2^{j+1}-1}$$

and

$$a(j + t, t + 1) = -\zeta^{2^{j+1}} \left(\sum_{k=1}^{j} \zeta^{-2^k} + \left\{ \sum_{k=j+1}^{j+t} \zeta^{-2^k} - \sum_{k=j+t+1}^{j+2t} \zeta^{-2^k} \right\} - \zeta^{-2^{j+1}} \right)$$

$$= 1 - \zeta^{2^{j+1}} \sum_{k=1}^{j} \zeta^{-2^k},$$

since the expression in braces vanishes in view of the fact that $2^k \equiv 2^{k+t} \pmod{q}$ for each k between $j + 1$ and $j + t$. Thus,

$$K(j + t, t + 1) = \begin{cases} K(j, 1), & \text{if } j \geq 1, \\ B^{-1}, & \text{if } j = 0. \end{cases}$$

This relation shows that G is non-free.

References

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