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MULTIDIMENSIONAL q-BETA INTEGRALS*

RONALD J. EVANS†

Abstract. A multidimensional extension of a q-beta integral of Andrews and Askey is evaluated. As an application, a short new proof of an important q-Selberg integral formula is given.

Key words. q-integral, Selberg integral, beta integrals

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1. Introduction. This paper has been motivated by Anderson's wonderfully innovative proof [2] of Selberg's multidimensional beta integral formula [17]. In § 2 (see Theorem 1), we present a new n-dimensional q-beta integral formula which reduces to that of Andrews and Askey [4, eqn. (2.2)] when $n = 1$ and that of Anderson [2, "claim"] when $q = 1$. Our proof is self-contained and in particular makes no appeal to the results of the aforementioned papers. In § 3, we apply Theorem 1 to give a surprisingly short, self-contained proof of the q-Selberg integral formula (1.8). Finally, we indicate in § 4 the modifications that can be made in § 3 to give a short proof of Kadell's extension of the q-Selberg integral formula containing the extra parameter m of Aomoto [5]; see Theorem 2. It is hoped that this method will lead to a short proof of a q-extension of the Selberg-Jack integral formula [15].

For some of the many applications and extensions of Selberg's integral, see the papers of Askey [6]-[8] and Kadell [14]-[16]. For character sum analogues of Selberg's integral, see the papers of Anderson [1], Evans [10] and van Wamelen [18].

Let

$$0 < q < 1, \tag{1.1}$$

and define, for complex x, α,

$$(\alpha)_\infty := \prod_{r=0}^{\infty} (1-\alpha q^r), \qquad (\alpha)_x := (\alpha)_\infty/(\alpha q^x)_\infty. \tag{1.2}$$

Define the q-gamma function

$$\Gamma_q(x) := (q)_{x-1}(1-q)^{1-x}, \qquad x \in \mathbb{C}. \tag{1.3}$$

As $q \to 1$, $\Gamma_q(x) \to \Gamma(x)$ [11, eqn. (1.10.3)]. For $\alpha, \beta \in \mathbb{C}$ and a (say) continuous function $f: \mathbb{C} \to \mathbb{C}$, define the q-integral

$$\int_\alpha^\beta f(x)\, d_qx := \int_0^\beta f(x)\, d_qx - \int_0^\alpha f(x)\, d_qx, \tag{1.4}$$

where

$$\int_0^\beta f(x)\, d_qx := (1-q) \sum_{m=0}^{\infty} f(\beta q^m) \beta q^m. \tag{1.5}$$

As $q \to 1$, $\int_\alpha^\beta f(x)\, d_qx \to \int_\alpha^\beta f(x)\, dx$ [11, p. 19]. For example, for $m > 0$,

$$\int_\alpha^\beta x^{m-1}\, d_qx = \frac{(\beta^m - \alpha^m)(1-q)}{(1-q^m)} \to \frac{\beta^m - \alpha^m}{m} \tag{1.6}$$

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† Department of Mathematics 0112, University of California, San Diego, La Jolla, California 92093-0112.

as $q\to 1$. The following q-integral extension of Euler's beta function integral is essentially a version of the q-binomial theorem [11, pp. 18–19]:

$$\int_0^1 t^{a-1}(tq)_{b-1}\,d_qt=\Gamma_q(a)\Gamma_q(b)/\Gamma_q(a+b),\qquad \mathrm{Re}\,(a), \mathrm{Re}\,(b)>0. \tag{1.7}$$

This is the case $n=1$ of the following n-dimensional q-Selberg integral formula [13, eqn. (4.18)]:

$$\begin{aligned} S_n(a,b,c)&:=\frac{1}{n!}\int_0^1\cdots\int_0^1\prod_{i=1}^n t_i^{a-1}(t_iq)_{b-1}\prod_{1\leqq i<j\leqq n}\prod_{k=1-c}^{c-1}(t_i-q^kt_j)\,d_qt_1\cdots d_qt_n\\ &=q^{ac\binom{n}{2}+2c^2\binom{n}{3}}\prod_{j=0}^{n-1}\frac{\Gamma_q(a+jc)\Gamma_q(b+jc)\Gamma_q(c+jc)}{\Gamma_q(a+b+(n-1+j)c)\Gamma_q(c)},\end{aligned}\tag{1.8}$$

where n, c are positive integers and Re (a), Re $(b)>0$. This reduces to Selberg's integral formula [17] when $q\to 1$. Note that the integrand in (1.8) is symmetric in the variables t_i. It is not difficult to show that the nonsymmetric version of (1.8) originally conjectured by Askey [6, Conj. 1] is equivalent to (1.8); see Kadell [13, p. 953]. Proofs of (1.8) have been given independently by Habsieger [12] and Kadell [14].

We observe here for later use that the value of the integral in (1.8) is unchanged if the upper limits of integration are replaced by q^{-u}, when u and b are integers such that $0\leqq u\leqq b-1$. This is because $(tq)_{b-1}$ vanishes for $t=q^{-1}, q^{-2},\cdots, q^{-u}$. It follows that the integral in (1.8) changes only by a factor of a power of q when the variables t_i are replaced by t_iq^{-u}.

2. Extension of the Andrews–Askey q-integral.

THEOREM 1. *Let u_i, s_i be integers such that*

$$0\leqq u_i\leqq s_i-1,\qquad i=0,1,\cdots,n, \tag{2.1}$$

and let z_i, w_i be complex variables with

$$w_i=z_iq^{-u_i},\qquad i=0,1,\cdots,n. \tag{2.2}$$

Then

$$\begin{aligned} L:=&\int_{t_n=w_{n-1}}^{w_n}\cdots\int_{t_2=w_1}^{w_2}\int_{t_1=w_0}^{w_1}\prod_{i=0}^n\prod_{j=1}^n\prod_{k=1}^{s_i-1}(z_i-q^kt_j)\\ &\cdot\prod_{1\leqq i<j\leqq n}(t_j-t_i)\,d_qt_1\,d_qt_2\cdots d_qt_n\\ &=(-1)^\sigma q^\tau\frac{\Gamma_q(s_0)\Gamma_q(s_1)\cdots\Gamma_q(s_n)}{\Gamma_q(s_0+s_1+\cdots+s_n)}\prod_{0\leqq i<j\leqq n}\prod_{k=1-s_j}^{s_i-1}(z_i-q^kz_j),\end{aligned}\tag{2.3}$$

where

$$\sigma=\sum_{i=1}^n is_i,\qquad \tau=\sum_{i=1}^n i\binom{s_i}{2}. \tag{2.4}$$

Remark 1. Suppose that all z_i are nonzero and all u_i are zero. Then the integral formula in Theorem 1 can be written in the form

$$\begin{aligned}&\int_{t_n=z_{n-1}}^{z_n}\cdots\int_{t_2=z_1}^{z_2}\int_{t_1=z_0}^{z_1}\prod_{i=0}^n\prod_{j=1}^n\left(\frac{qt_j}{z_i}\right)_{s_i-1}\\ &\cdot\prod_{1\leqq i<j\leqq n}(t_j-t_i)\,d_qt_1\,d_qt_2\cdots d_qt_n\\ &=\frac{\Gamma_q(s_0)\Gamma_q(s_1)\cdots\Gamma_q(s_n)}{\Gamma_q(s_0+s_1+\cdots+s_n)}\prod_{0\leqq i<j\leqq n}z_j\left(\frac{z_i}{z_j}\right)_{s_j}\left(\frac{qz_j}{z_i}\right)_{s_i-1}.\end{aligned}\tag{2.5}$$

Since (2.5) is valid for all positive integers s_i by Theorem 1, it follows by analytic continuation (cf. [3, p. 115]) that it holds for all complex s_i with

$$\operatorname{Re}(s_i) > \max_{0\leqq j\leqq n} \frac{\log|z_j/z_i|}{|\log q|}, \qquad i=0,1,2,\cdots,n.$$

If $n=1$, (2.5) reduces to the Andrews–Askey q-integral [4, (2.2)].

Remark 2. From (2.5) and [9, Thm. 2.2], it may be deduced that the constant term of the Laurent polynomial

$$P(z_1,\cdots,z_n) := \int_0^1 \cdots \int_0^1 \prod_{1\leqq i,j\leqq n} (qt_jz_j/z_i)_{s_i-1} \cdot \prod_{1\leqq i<j\leqq n} (t_j - t_iz_i/z_j)\, d_qt_1 \cdots d_qt_n$$

equals

$$\prod_{i=1}^{n} (1-q)/(1-q^{s_i+s_{i+1}+\cdots+s_n}). \tag{2.6}$$

It would be interesting to find a proof independent of [9].

Proof of Theorem 1. Assume that each z_i is an integral power of q and that the sequence $w_0, w_1, w_2, \cdots, w_n$ is monotone. It suffices to prove (2.3) under these assumptions, since both sides of (2.3) are polynomials in $z_0, \cdots, z_n$.

Consider any one of the rightmost factors in (2.3), say

$$z_\alpha - q^\gamma z_\beta, \tag{2.7}$$

with

$$0 \leqq \alpha < \beta \leqq n, \qquad 1 - s_\beta \leqq \gamma \leqq s_\alpha - 1. \tag{2.8}$$

We will show that $z_\alpha - q^\gamma z_\beta$ is also a factor of L by showing that L vanishes under the assumption

$$z_\alpha = q^\gamma z_\beta. \tag{2.9}$$

The q-integral L is a series by definition, and it suffices to show that each summand in this series vanishes. This will be accomplished if we can show

$$\prod_{k=1}^{s_\alpha-1} (z_\alpha - q^k t) \prod_{m=1}^{s_\beta-1} (z_\beta - q^m t) = 0 \quad \text{for all } t \in S, \tag{2.10}$$

where S is the set of integral powers of q between w_α and w_β including $\max(w_\alpha, w_\beta)$ but not $\min(w_\alpha, w_\beta)$. Define

$$A = \{z_\alpha q^{-k} : 1 \leqq k \leqq s_\alpha - 1\}, \qquad B = \{z_\beta q^{-m} : 1 \leqq m \leqq s_\beta - 1\}. \tag{2.11}$$

Since $z_\alpha = q^\gamma z_\beta$ by (2.9), there is no integral power of q lying strictly between the sets A and B on the real axis. It is thus seen that $A \cup B \supset S$, and (2.10) follows. We have now proved that L is divisible by each of the linear factors in (2.7), and hence by the polynomial

$$\prod_{0\leqq i<j\leqq n} \prod_{k=1-s_j}^{s_i-1} (z_i - q^k z_j). \tag{2.12}$$

By definition of L, if we view L as a polynomial in z_0 with leading term $C_n z_0^\nu$ (with C_n independent of z_0), then

$$\nu = n(s_0 - 1) + (s_1 + \cdots + s_n). \tag{2.13}$$

Viewing (2.12) as a polynomial in z_0, we see that it also has degree ν. Thus it remains to prove that

$$C_n=(-1)^{\sigma}q^{\tau}\frac{\Gamma_q(s_0)\cdots\Gamma_q(s_n)}{\Gamma_q(s_0+\cdots+s_n)}\prod_{1\leqq i<j\leqq n}\prod_{k=1-s_j}^{s_i-1}(z_i-q^kz_j). \tag{2.14}$$

First consider the case $n=1$. Then C_1 is the coefficient of $z_0^{s_0+s_1-1}$ in

$$\int_{t=w_0}^{w_1}\prod_{k=1}^{s_0-1}(z_0-q^kt)\prod_{m=1}^{s_1-1}(z_1-q^mt)\,d_qt, \tag{2.15}$$

so C_1 is the coefficient of $z_0^{s_0+s_1-1}$ in

$$-\prod_{m=1}^{s_1-1}(-q^m)\int_{w_1}^{w_0}t^{s_1-1}z_0^{s_0-1}(qt/z_0)_{s_0-1}\,d_qt. \tag{2.16}$$

Replace t by z_0t to see that C_1 is the constant term in the expansion in z_0 of

$$(-1)^{s_1}q^{\binom{s_1}{2}}\int_{w_1/z_0}^{q^{-u_0}}t^{s_1-1}(qt)_{s_0-1}\,d_qt. \tag{2.17}$$

The constant term in (2.17) is unchanged if the lower limit of q-integration is replaced by 0. It is further unchanged if the upper limit of q-integration is replaced by 1, since

$$(qt)_{s_0-1}=0\quad\text{for}\quad t=q^{-i}\qquad(i=1,2,\cdots,s_0-1). \tag{2.18}$$

It now follows from (1.7) that (2.14) holds for $n=1$, so the proof of Theorem 1 is complete in the case $n=1$.

Suppose now that $n>1$ and that Theorem 1 holds with $(n-1)$ in place of n. Directly from (2.3), we see that C_n is the coefficient of $z_0^{(s_0+\cdots+s_n)-1}$ in

$$\begin{aligned}&\int_{t_n=w_{n-1}}^{w_n}\cdots\int_{t_2=w_1}^{w_2}\prod_{i=1}^{n}\prod_{j=2}^{n}\prod_{k=1}^{s_i-1}(z_i-q^kt_j)\cdot\prod_{2\leqq i<j\leqq n}(t_j-t_i)\\&\cdot(-1)^{s_1+\cdots+s_n}q^{\binom{s_1}{2}+\cdots+\binom{s_n}{2}}\int_{t=w_1}^{w_0}t^{(s_1+\cdots+s_n)-1}\\&\cdot\prod_{k=1}^{s_0-1}(z_0-q^kt)\,d_qt\,d_qt_2\cdots d_qt_n.\end{aligned} \tag{2.19}$$

The inner integral on t in (2.19) may be replaced by

$$z_0^{(s_0+\cdots+s_n)-1}\int_{w_1/z_0}^{q^{-u_0}}t^{(s_1+\cdots+s_n)-1}(qt)_{s_0-1}\,d_qt, \tag{2.20}$$

and just as with (2.17), the desired coefficient is unchanged if we further replace the lower and upper limits of q-integration in (2.20) by zero and 1, respectively. Thus by (1.7), C_n is the constant term of the polynomial in z_0 obtained from (2.19) by replacing the inner integral on t by

$$\frac{\Gamma_q(s_0)\Gamma_q(s_1+\cdots+s_n)}{\Gamma_q(s_0+s_1+\cdots+s_n)}. \tag{2.21}$$

By induction on n, the proof of Theorem 1 is complete.

3. Proof of the q-Selberg integral formula. In this section we apply Theorem 1 to give a short proof of the q-Selberg integral formula (1.8). The result is true for $n=1$ by (1.7), so let $n>1$. We may assume that a and b are positive integers, as the result can be extended by analytic continuation to hold whenever Re (a), Re $(b)>0$.

Given polynomials

$$E(t)=\prod_{i=1}^{n}(t-e_i),\qquad H(t)=\prod_{i=1}^{n-1}(t-h_i) \tag{3.1}$$

with

$$0\leqq e_1\leqq h_1\leqq e_2\leqq h_2\leqq\cdots\leqq h_{n-1}\leqq e_n\leqq 1, \tag{3.2}$$

use for brevity the symbolic notation

$$\int_{E\in D_n}\{\ \}\,d_qE:=\int_{e_n=0}^{1}\cdots\int_{e_2=0}^{e_3}\int_{e_1=0}^{e_2}\{\ \}\cdot\prod_{1\leqq i<j\leqq n}(e_i-e_j)\,d_qe_1\,d_qe_2\cdots d_qe_n \tag{3.3}$$

and

$$\int_{H\in D_{n-1}(E)}\{\ \}\,d_qH:=\int_{h_{n-1}=e_{n-1}}^{e_n}\cdots\int_{h_2=e_2}^{e_3}\int_{h_1=e_1}^{e_2}\{\ \}\cdot\prod_{1\leqq i<j\leqq n-1}(h_i-h_j)\,d_qh_1\,d_qh_2\cdots d_qh_{n-1}. \tag{3.4}$$

Note that

$$\int_{E\in D_n}\int_{H\in D_{n-1}(E)}=\int_{H\in D_{n-1}}\int_{E\in D_n(V)}, \tag{3.5}$$

where

$$V(t)=\prod_{i=0}^{n}(t-v_i)\quad\text{with } v_0=0,\quad v_n=1,\quad v_i=qh_i\quad(1\leqq i\leqq n-1). \tag{3.6}$$

Define

$$I_n(a,b,c):=\int_{E\in D_n}\int_{H\in D_{n-1}(E)}\prod_{i=1}^{n}e_i^{a-1}(qe_i)_{b-1}\cdot\prod_{i=1}^{n}\prod_{j=1}^{n-1}\prod_{k=1}^{c-1}(q^{c-1}e_i-q^kh_j)\,d_qH\,d_qE. \tag{3.7}$$

If we replace n by $n-1$ in Theorem 1 and then further take $t_i=h_i$, $s_i=c$, $u_i=c-1$, $w_i=e_{i+1}$, $z_i=q^{c-1}e_{i+1}$, then Theorem 1 yields

$$\begin{aligned}\int_{H\in D_{n-1}(E)}\prod_{i=1}^{n}\prod_{j=1}^{n-1}\prod_{k=1}^{c-1}(q^{c-1}e_i-q^kh_j)\,d_qH&=(-1)^{\binom{n-1}{2}+c\binom{n}{2}}q^{\binom{n}{2}\binom{c}{2}}\frac{\Gamma_q(c)^n}{\Gamma_q(cn)}\\&\cdot\prod_{1\leqq i<j\leqq n}\prod_{k=1-c}^{c-1}(q^{c-1}e_i-q^{k+c-1}e_j).\end{aligned} \tag{3.8}$$

Thus, by definition of $S_n(a,b,c)$ and $I_n(a,b,c)$,

$$I_n(a,b,c)=(-1)^{\binom{n-1}{2}+c\binom{n}{2}}q^{\binom{n}{2}\binom{c}{2}+\binom{n}{2}\binom{2c-1}{2}}\frac{\Gamma_q(c)^n}{\Gamma_q(cn)}S_n(a,b,c). \tag{3.9}$$

By (3.5) and (3.6), interchange of integration in (3.7) yields

$$
\begin{aligned}
I_n(a,b,c)=\int_{H\in D_{n-1}}\int_{E\in D_n(V)} & (-1)^{n(a-1)}q^{-n\binom{a}{2}}\prod_{j=1}^{n}\prod_{k=1}^{a-1}(0-q^ke_j)\\
(3.10)\qquad & \cdot\prod_{j=1}^{n}\prod_{k=1}^{b-1}(1-q^ke_j)\\
& \cdot q^{2\binom{n}{2}\binom{c-1}{2}}\prod_{i=1}^{n-1}\prod_{j=1}^{n}\prod_{k=1}^{c-1}(v_i-q^ke_j)\,d_qE\,d_qH.
\end{aligned}
$$

Apply Theorem 1 with $t_i=e_i$, $s_0=a$, $s_n=b$, $s_i=c$ $(1\leqq i\leqq n-1)$, $u_i=0$, $w_i=v_i$, and $z_i=v_i$ to see that the inner integral on E equals

$$
\begin{aligned}
& (-1)^{\binom{n-1}{2}+c\binom{n}{2}}q^{\binom{n}{2}\binom{c}{2}+2\binom{n}{2}\binom{c-1}{2}}\\
(3.11)\qquad & \cdot\frac{\Gamma_q(a)\Gamma_q(b)\Gamma_q(c)^{n-1}}{\Gamma_q(a+b+(n-1)c)}\prod_{j=1}^{n-1}v_j^{a+c-1}\prod_{j=1}^{n-1}\prod_{k=1-c}^{b-1}(1-q^kv_j)\\
& \cdot\prod_{1\leqq i<j\leqq n-1}\prod_{k=1-c}^{c-1}(v_i-q^kv_j).
\end{aligned}
$$

Before integrating (3.11) on H, make the change of variables $h_i\to q^{c-1}h_i$ (so $v_i\to q^cv_i$). As a result,

$$
\begin{aligned}
I_n(a,b,c)= & (-1)^{\binom{n-1}{2}+c\binom{n}{2}}\\
(3.12)\qquad & \cdot q^{\binom{n}{2}\binom{c}{2}+2\binom{n}{2}\binom{c-1}{2}+(c-1)\binom{n}{2}+\binom{n-1}{2}\binom{2c}{2}+c(a+c-1)(n-1)}\\
& \cdot\frac{\Gamma_q(a)\Gamma_q(b)\Gamma_q(c)^{n-1}}{\Gamma_q(a+b+(n-1)c)}S_{n-1}(a+c,b+c,c).
\end{aligned}
$$

Comparison of (3.9) and (3.12) yields

$$
\begin{aligned}
(3.13)\qquad S_n(a,b,c)= & q^{ac(n-1)+c^2\binom{n-1}{2}}\frac{\Gamma_q(a)\Gamma_q(b)\Gamma_q(cn)}{\Gamma_q(a+b+(n-1)c)\Gamma_q(c)}\\
& \cdot S_{n-1}(a+c,b+c,c)
\end{aligned}
$$

and the result follows by induction on n. □

4. Extension of the q-Selberg integral. Let $S_{n,m}(a,b,c)$ denote the extension of the q-Selberg integral $S_n(a,b,c)$ obtained by inserting the factor $t_1t_2\cdots t_m$ in the integrand in (1.8), where $0\leqq m\leqq n$. In Theorem 2 below, we evaluate $S_{n,m}(a,b,c)$. It is not difficult to show that Theorem 2 is equivalent to the case $l=0$ of [14, Thm. 2]; see [14, eqns. (4.17), (4.19)].

THEOREM 2. *For positive integers n, c and* $\mathrm{Re}(a)$, $\mathrm{Re}(b)>0$,

$$
(4.1)\qquad S_{n,m}(a,b,c)=\frac{S_n(a,b,c)T_{n,m}(a,b,c)}{\binom{n}{m}},
$$

where

$$
(4.2)\qquad T_{n,m}(a,b,c):=q^{c\binom{m}{2}}\prod_{i=n-m}^{n-1}\frac{(1-q^{a+ci})(1-q^{c+ci})}{(1-q^{a+b+c(n-1+i)})(1-q^{cn-ci})}.
$$

Proof. We proceed as in the proof in § 3, with the following modifications. Let u be an indeterminate and let $S_n(a, b, c, u)$ be the extension of the q-Selberg integral $S_n(a, b, c)$ obtained by inserting the factor $\prod_{i=1}^{n}(u-t_i)$ in the integrand of (1.8). We must show that

$$\frac{S_n(a, b, c, u)}{S_n(a, b, c)} = \sum_{m=0}^{n} (-1)^m T_{n,m}(a, b, c)u^{n-m}. \tag{4.3}$$

Let $I_n(a, b, c, u)$ be the extension of $I_n(a, b, c)$ obtained by inserting the factor $q^{c(n-1)}H(u/q)$ in the integrand in (3.7). By Lagrange interpolation,

$$q^{c(n-1)}H\left(\frac{u}{q}\right) = \sum_{r=1}^{n} q^{c(n-1)}H\left(\frac{e_r}{q}\right) \prod_{i\neq r} \frac{u-e_i}{e_r-e_i}, \tag{4.4}$$

for distinct e_i. Thus, from (3.7),

$$\begin{aligned} I_n(a, b, c, u) = &\int_{E\in D_n} \sum_{r=1}^{n} \prod_{i\neq r} \frac{u-e_i}{e_r-e_i} \prod_{i=1}^{n} e_i^{a-1}(qe_i)_{b-1} \\ &\cdot \int_{H\in D_{n-1}(E)} \prod_{i=1}^{n} \prod_{j=1}^{n-1} \prod_{k=1}^{c-1+\delta(i,r)} (q^{c-1}e_i - q^k h_j)\, d_qH\, d_qE, \end{aligned} \tag{4.5}$$

where $\delta(i, r) = 1$ if $i = r$ and $\delta(i, r) = 0$ if $i \neq r$. If for each fixed r we replace n by $n-1$ in Theorem 1, and then further take $t_i = h_i$, $s_i = c + \delta(i, r)$, $u_i = c-1$, $w_i = e_{i+1}$, and $z_i = q^{c-1}e_{i+1}$, then Theorem 1 shows that the inner integral on H in (4.5) equals

$$\text{RHS (3.8)}\ q^{(n-1)(2c-1)} \frac{(1-q^c)}{(1-q^{cn})} \prod_{i\neq r} (q^{-c}e_r - e_i), \tag{4.6}$$

where RHS (3.8) denotes the right-hand side of (3.8). Thus

$$\begin{aligned} I_n(a, b, c, u) = &\, q^{(n-1)(2c-1)} \frac{(1-q^c)}{(1-q^{cn})} \int_{E\in D_n} \text{RHS (3.8)} \\ &\cdot \prod_{i=1}^{n} e_i^{a-1}(qe_i)_{b-1} \sum_{r=1}^{n} \prod_{i\neq r} \frac{u-e_i}{e_r-e_i} (q^{-c}e_r - e_i)\, d_qE. \end{aligned} \tag{4.7}$$

Given a polynomial $F(u)$, let $F^*(u)$ denote its q^{-c}-derivative [11, p. 22], namely

$$F^*(u) = \frac{F(u) - F(q^{-c}u)}{u - q^{-c}u}. \tag{4.8}$$

Since

$$E^*(e_r) = \prod_{i\neq r} (q^{-c}e_r - e_i), \tag{4.9}$$

the inner sum on r in (4.7) equals $E^*(u)$. Thus

$$I_n(a, b, c, u) = \text{RHS (3.9)}\ q^{(n-1)(2c-1)} \frac{(1-q^c)}{(1-q^{cn})} \frac{S_n^*(a, b, c, u)}{S_n(a, b, c)}. \tag{4.10}$$

After interchanging the order of integration, we obtain

$$I_n(a, b, c, u) = \text{RHS (3.12)}\ q^{(n-1)(2c-1)} \frac{S_{n-1}(a+c, b+c, c, uq^{-c})}{S_{n-1}(a+c, b+c, c)}. \tag{4.11}$$

Comparing (4.10) and (4.11), we arrive at the "differential equation"

$$\frac{S_n^*(a, b, c, u)}{S_n(a, b, c)} = \frac{1-q^{cn}}{1-q^c} \frac{S_{n-1}(a+c, b+c, c, uq^{-c})}{S_{n-1}(a+c, b+c, c)}. \tag{4.12}$$

By induction on n, (4.3) furnishes a solution to (4.12). Moreover, (4.3) is valid for $u=0$, by (1.8) with $a+1$ in place of a. Hence (4.3) is proved.

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