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A Fundamental Region for Hecke's Modular Group

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Hecke proved analytically that when $\lambda \geqslant 2$ or when $\lambda = 2\cos(\pi/q)$, $q \in Z$, $q \geqslant 3$, then $B(\lambda) = \{\tau: \operatorname{Im} \tau > 0, |\operatorname{Re} \tau| < \lambda/2, |\tau| > 1\}$ is a fundamental region for the group $G(\lambda) = \langle S_\lambda, T\rangle$, where $S_\lambda: \tau \to \tau + \lambda$ and $T: \tau \to -1/\tau$. He also showed that $B(\lambda)$ fails to be a fundamental region for all other $\lambda > 0$ by proving that $G(\lambda)$ is not discontinuous. We give an elementary proof of these facts and prove a related result concerning the distribution of $G(\lambda)$-equivalent points.

For each $\lambda > 0$, let $G(\lambda)$ be the group generated by the transformations $S_\lambda: \tau \to \tau + \lambda$ and $T: \tau \to -1/\tau$ defined on $H = \{\tau: \operatorname{Im} \tau > 0\}$. Let $B(\lambda) = \{\tau \in H: |\operatorname{Re} \tau| < \lambda/2, |\tau| > 1\}$. Let Z denote the integers. Hecke [1, pp. 11–20; 2, pp. 599–616] proved analytically that $B(\lambda)$ is a fundamental region (as defined in [3, p. 22]) for $G(\lambda)$ when $\lambda \geqslant 2$ or when $\lambda = 2\cos(\pi/q)$ for some $q \in Z$, $q \geqslant 3$ (in the latter case we write $\lambda \in C$). We give an elementary proof of this fact. When $0 < \lambda < 2$, $\lambda \notin C$, Hecke [2, pp. 609, 613–614] proved that $G(\lambda)$ is not discontinuous (so that there can be no fundamental region for $G(\lambda)$). We present here a slightly simplified version of his proof and show, moreover, that for any $\tau \in H$, the set of all points $G(\lambda)$-equivalent to τ is dense in H.

THEOREM 1. *Each $\gamma \in H$ is $G(\lambda)$-equivalent to a point in $\overline{B(\lambda)}$, (the closure of $B(\lambda)$).*

Proof. Define the following transformations on H:

$T_1: \tau \to \tau/|\tau|^2$ (reflection in the unit circle),

$T_2: \tau \to -\bar{\tau}$ (reflection in the line $\operatorname{Re} \tau = 0$),

$T_3: \tau \to -(\bar{\tau} + \lambda)$ (reflection in the line $\operatorname{Re} \tau = -\lambda/2$).

Since $S_\lambda = T_2 T_3$ and $T = T_1 T_2$, it is easily seen that $G(\lambda)$ consists of the words in $\langle T_1, T_2, T_3 \rangle$ of even length. Hence, it suffices to find $V \in \langle T_1, T_2, T_3 \rangle$ such that $V\gamma \in \overline{B(\lambda)}$, for if $V \notin G(\lambda)$, then $T_2 V \in G(\lambda)$.

Define a sequence of points $\tau_n = x_n + iy_n$ inductively as follows: apply T_2 and T_3, if necessary, to move γ horizontally to a point τ_1 in the strip $E_\lambda = \{\tau \in H: -\lambda/2 \leqslant \operatorname{Re} \tau \leqslant 0\}$. Given τ_n $(n \geqslant 1)$, apply T_2 and T_3 to move $T_1 \tau_n$ horizontally to a point $\tau_{n+1} \in E_\lambda$. We will assume that $|\tau_n| < 1$ for each n, otherwise the theorem is proved. Thus, $y_{n+1} = y_n / |\tau_n|^2 > y_n$. Let w be a cluster point of $\{\tau_n\}$. Note $\operatorname{Im} w > 0$. If $|w| < 1$, then $\{\tau_n\}$ has an infinite subsequence $\{\tau_{n_k}\}$ such that $|\tau_{n_k}| \leqslant c < 1$, so that $y_{n_k} \geqslant y_{n_1}/c^{2(k-1)} \to \infty$ as $k \to \infty$, a contradiction. Hence, $|w| = 1$. When $\lambda < 2$, let v denote the point of intersection between the unit circle and the line $\operatorname{Re} \tau = -\lambda/2$. We will assume that $\lambda < 2$ and that $w = v$ is the unique cluster point of $\{\tau_n\}$, otherwise $T_1 \tau_n \in B(\lambda)$ for some large n. If $\arg \tau_n \leqslant \arg v$ for some n, then $\operatorname{Im} \tau_{n+1} > \operatorname{Im} v$, contradicting the fact that $y_n \uparrow \operatorname{Im} v$. Hence, $\arg \tau_n > \arg v$ for each n. Now there exists an N such that for all $n \geqslant N$, $\tau_{n+1} = T_3 T_1 \tau_n$, so that $x_{n+1} = -\lambda - x_n/(x_n^2 + y_n^2)$. Let $n \geqslant N$. Note that $x_n < 0$, since $x_{n+1} \geqslant -\lambda/2 > -\lambda$. Letting $\pi\theta = \pi - \arg v$ (so that $\lambda = 2 \cos \pi\theta$), we have

$$\begin{aligned} x_{n+1} - x_n &= \frac{1}{x_n}\left(-\lambda x_n - \frac{x_n^2}{x_n^2 + y_n^2} - x_n^2\right) \\ &= -\frac{1}{x_n}(\lambda x_n + \cos^2(\arg \tau_n) + x_n^2) \\ &> -\frac{1}{x_n}(\lambda x_n + \cos^2(\arg v) + x_n^2) \\ &= -\frac{1}{x_n}(x_n + \cos \pi\theta)^2 \geqslant 0. \end{aligned}$$

Thus, $x_{n+1} > x_n$ for each $n \geqslant N$, which contradicts the fact that $x_n \to \operatorname{Re} v$. ∎

Thus, $B(\lambda)$ is a fundamental region for $G(\lambda)$ if and only if no two distinct points of $B(\lambda)$ are $G(\lambda)$-equivalent. We now show this is the case when $\lambda \geqslant 2$ or $\lambda \in C$.

THEOREM 2. *When $\lambda \geqslant 2$, no two distinct points of $B(\lambda)$ are $G(\lambda)$-equivalent.*

Proof. Choose $V \neq I$ (I is the identity) in $G(\lambda)$ and $\tau \in B(\lambda)$. We will show that $V\tau \notin B(\lambda)$. We can write V in the form $V = S_\lambda^{k_r} T S_\lambda^{k_{r-1}} T \cdots S_\lambda^{k_2} T S_\lambda^{k_1}$, where $r \geqslant 1$, each $k_i \in Z$, and $k_i \neq 0$ if $2 \leqslant i \leqslant r - 1$. Let $\tau_i =$

$TS_\lambda^{k_i}TS_\lambda^{k_{i-1}} \cdots TS_\lambda^{k_1}\tau$. It is easily seen that $|\tau_i| < 1$ for $1 \leqslant i \leqslant r-1$. Thus, $V\tau = S_\lambda^{k_r}\tau_{r-1} \notin B(\lambda)$. ∎

In order to handle the case $\lambda \in C$, we shall need two lemmas. Whenever $\lambda \in C$, we shall write $\lambda = 2\cos(\pi/q)$, where $q \in Z$, $q \geqslant 3$.

LEMMA 1. *When $\lambda \in C$, no two points of $B(\lambda)$ are equivalent under a nonidentity transformation in $\langle T_1, T_3\rangle$.*

Proof. If the lemma is false, then there exist points $\tau, \tau' \in B(\lambda)$ with, say, $\operatorname{Im}\tau' \geqslant \operatorname{Im}\tau$ and a word $V \neq I$ in $\langle T_1, T_3\rangle$ such that $V\tau = \tau'$. Note $V \neq T_3$, as $T_3\tau \notin B(\lambda)$. Hence, as T_1 and T_3 have order 2, V can have either the form $T_3^\alpha(T_1T_3)^n$ or $T_3^\alpha(T_3T_1)^n$, where $n \in Z$, $n \neq 0$, and $\alpha = 0$ or 1. If V has the latter form, then $V = T_3^\alpha(T_1T_3)^{-n}$ because $T_3T_1 = (T_1T_3)^{-1}$. Thus, in any case V has the former form. Now for all $n \in Z$, $(T_1T_3)^n$ is the linear fractional transformation with matrix

$$\begin{pmatrix} a_n & b_n \\ c_n & d_n \end{pmatrix} = \frac{1}{\sin \pi\theta}\begin{pmatrix} \sin \pi\theta(1-n) & -\sin \pi\theta n \\ \sin \pi\theta n & \sin \pi\theta(n+1) \end{pmatrix}$$

Since $(T_1T_3)^q = I$, we may write $V = T_3^\alpha(T_1T_3)^n$, where $\alpha = 0$ or 1, $n \in Z$, $1 \leqslant n \leqslant q-1$. Write $\tau = x + iy$. As $c_nd_n \geqslant 0$, we have

$$|c_n\tau + d_n|^2 = c_n^2|\tau|^2 + d_n^2 + 2c_nd_nx > c_n^2 + d_n^2 - \lambda c_nd_n = 1,$$

so that

$$\operatorname{Im}\tau' = \operatorname{Im}(T_1T_3)^n\tau = \frac{y}{|c_n\tau + d_n|^2} < y = \operatorname{Im}\tau,$$

a contradiction. ∎

LEMMA 2. *Let $\lambda \in C$, let $x + iy = \tau \in H$, and let $W \in \langle T_1, T_3\rangle$, $W \neq I$, $W \neq T_1$. If either*

(i) $\operatorname{Re}\tau > 0$

or

(ii) $\tau \in B(\lambda)$,

then $\operatorname{Re} W\tau < 0$.

Proof. We can write W in the form $W = T_1^\alpha(T_1T_3)^n$, where $\alpha = 0$ or 1, $n \in Z$, $1 \leqslant n \leqslant q-1$. To show that $\operatorname{Re} W\tau < 0$, it suffices to show that $\operatorname{Re}(T_1T_3)^n\tau < 0$. We have (in the notation of the previous lemma)

$$\operatorname{Re}(T_1T_3)^n\tau = \frac{(a_nx + b_n)(c_nx + d_n) + a_nc_ny^2}{|c_n\tau + d_n|^2}.$$

Note that $a_n \leqslant 0$, $b_n \leqslant 0$, $c_n \geqslant 0$, and $d_n \geqslant 0$. Hence, if (i) holds, $a_n c_n y^2 \leqslant 0$ and $(a_n x + b_n)(c_n x + d_n) < 0$, so $\mathrm{Re}(T_1 T_3)^n \tau < 0$. If (ii) holds, then

$$\begin{aligned} \mathrm{Re}(T_1T_3)^n \tau &= \frac{b_n d_n + a_n c_n \mid \tau \mid^2 + (a_n d_n + b_n c_n)\, x}{\mid c_n \tau + d_n \mid^2} \\ &\leqslant \frac{b_n d_n + a_n c_n + (a_n d_n + b_n c_n)(-\lambda/2)}{\mid c_n \tau + d_n \mid^2} \\ &= \frac{-\cos(\pi/q)}{\mid c_n \tau + d_n \mid^2} < 0. \end{aligned}$$

∎

THEOREM 3. *If $\lambda \in C$, no two distinct points of $B(\lambda)$ are $G(\lambda)$-equivalent.*

Proof. It suffices to show that no two points of $B(\lambda)$ are equivalent under a transformation $V \in \langle T_1, T_2, T_3 \rangle$, where $V \neq I$, $V \neq T_2$. If the contrary is true, choose a word v for V in $\langle T_1, T_2, T_3 \rangle$ of minimal length L for which $V \neq T_2$, $V \neq I$, and there exists $\tau \in B(\lambda)$ such that $V\tau \in B(\lambda)$. By Lemma 1, such a word must contain T_2. No word for V of length L can begin or end with T_2. For if $V = T_2 Y$, then $Y \neq T_2$, $Y \neq I$, and $Y\tau \in B(\lambda)$, which contradicts the minimality of L; similarly, if $V = YT_2$, then $Y \neq T_2$, $Y \neq I$, and $Y(T_2\tau) \in B(\lambda)$, a contradiction. Thus, $v = W_1 T_2 W_2 T_2 \cdots W_k T_2 W_{k+1}$ $(k \geqslant 1)$, where $I \neq W_i \in \langle T_1, T_3 \rangle$ for each i. Moreover, for each i, $W_i \neq T_1$. For if W_1 or W_{k+1} equals T_1, then since $T_1 T_2 = T_2 T_1$, V would equal a word of length L which begins or ends with T_2; if $W_i = T_1$ for some i such that $2 \leqslant i \leqslant k$, then since $T_2 T_1 T_2 = T_1$, V would equal a word of length smaller than L.

Let $\tau_i = T_2 W_i T_2 W_{i+1} \cdots T_2 W_{k+1} \tau$. We will show by induction on i that $\mathrm{Re}\, \tau_i < 0$, $(2 \leqslant i \leqslant k+1)$. Since $V\tau \in B(\lambda)$, $\mathrm{Re}\, \tau_2 = \mathrm{Re}\, W_1^{-1} V \tau < 0$ by Lemma 2. Assume $\mathrm{Re}\, \tau_m < 0$ for an m such that $2 \leqslant m \leqslant k$. Then $\mathrm{Re}\, T_2 \tau_m > 0$, so by Lemma 2, $\mathrm{Re}\, \tau_{m+1} = \mathrm{Re}\, W_m^{-1} T_2 \tau_m < 0$, completing the induction. As $\tau \in B(\lambda)$, $\mathrm{Re}\, W_{k+1}\tau < 0$ by Lemma 2. Hence, $\mathrm{Re}\, \tau_{k+1} = \mathrm{Re}\, T_2 W_{k+1} \tau > 0$, a contradiction. ∎

We now investigate the distribution of $G(\lambda)$-equivalent points in H when $0 < \lambda < 2$, $\lambda \notin C$.

LEMMA 3. *Let*

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

be the matrix of the linear fractional transformation $W \in G(\lambda)$. Then W has a fixed point in H if and only if $\mid a + d \mid < 2$.

Proof. $W\tau = \tau$ if and only if $\tau = \{a - d \pm \sqrt{(d+a)^2 - 4}\}/2c$. ▮

LEMMA 4. *Suppose $W \in G(\lambda)$ has infinite order and W has a fixed point $\tau_1 \in H$. Let $t(\tau) = (\tau - \tau_1)/(\tau - \bar{\tau}_1)$, where $\bar{\tau}_1$ is the complex conjugate of τ_1. Then for each $\tau \in H - \{\tau_1\}$, the set $J_\tau = \{W^n\tau : n \in Z\}$ is dense on the circle $K_\tau = \{\sigma : | t(\sigma)| = | t(\tau)|\}$.*

Proof. Let

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

be the matrix of W. Note that $\rho = c\tau_1 + d$ is the characteristic value of

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

corresponding to the characteristic vector $\binom{\tau_1}{1}$.

Since ρ and $\bar{\rho}$ are the roots of the characteristic equation

$$x^2 - (a+d)x + 1 = 0,$$

we have $\rho\bar{\rho} = 1$. Now for any τ, $t(W\tau) = (W\tau - W\tau_1)/(W\tau - W\bar{\tau}_1)$ since τ_1 and $\bar{\tau}_1$ are fixed by W. Thus,

$$t(W\tau) = \frac{\tau - \tau_1}{(c\tau + d)(c\tau_1 + d)} \Big/ \frac{\tau - \bar{\tau}_1}{(c\tau + d)(c\bar{\tau}_1 + d)} = \frac{\bar{\rho}}{\rho}\, t(\tau) = \rho^{-2} t(\tau).$$

Thus, for all $n \in Z$, $t(W^n\tau) = \rho^{-2n} t(\tau)$. Since τ_1 is nonreal and W has infinite order,

$$\begin{pmatrix} \tau_1 \\ 1 \end{pmatrix} \neq \begin{pmatrix} a & b \\ c & d \end{pmatrix}^n \begin{pmatrix} \tau_1 \\ 1 \end{pmatrix} = \rho^n \begin{pmatrix} \tau_1 \\ 1 \end{pmatrix}, \quad \text{for each } n \geqslant 1.$$

Otherwise, writing

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix}^n = \begin{pmatrix} a^{(n)} & b^{(n)} \\ c^{(n)} & d^{(n)} \end{pmatrix},$$

we would have $(a^{(n)} - 1)\,\tau_1 = -b^{(n)}$ and $c^{(n)}\tau_1 = 1 - d^{(n)}$, so that $a^{(n)} = d^{(n)} = 1$ and $b^{(n)} = c^{(n)} = 0$, a contradiction.

Therefore, ρ is not a root of unity, and, consequently, $\{t(W^n\tau) : n \in Z\}$ is dense on the circle $\{z : | z | = | t(\tau)|\}$. Thus, J_τ is dense on K_τ. ▮

LEMMA 5. *If $0 < \lambda < 2$, $\lambda \notin C$, then there exists a $W \in G(\lambda)$ such that W has infinite order and W has a fixed point in H.*

Proof. *Case* 1. θ is irrational. Choose $W = TS_\lambda$ so that W has matrix

$$\begin{pmatrix} a_1 & b_1 \\ c_1 & d_1 \end{pmatrix}$$

(in the notation of the proof of Lemma 1). By Lemma 3, W has a fixed point in H. Since θ is irrational, $c_n \neq 0$ for all $n \geqslant 1$. Thus W has infinite order.

Case 2. $\theta = p/q$, $(p, q) = 1$, $2 \leqslant p < q/2$. Choose $W = T(TS_\lambda)^k$, where $kp \equiv 1 \pmod q$. Note that W has matrix

$$\begin{pmatrix} -c_k & -d_k \\ a_k & b_k \end{pmatrix} = \begin{pmatrix} -c_k & -d_k \\ a_k & -c_k \end{pmatrix}.$$

Since

$$| c_k | = \left| \frac{\sin(\pi pk/q)}{\sin(\pi p/q)} \right| = \frac{\sin(\pi/q)}{\sin(\pi p/q)} < 1,$$

W has a fixed point by Lemma 3.

To show that W has infinite order, we will show that

$$\begin{pmatrix} -c_k & -d_k \\ a_k & -c_k \end{pmatrix}$$

has a characteristic value ρ which is not a root of unity. Let c_k' be any algebraic conjugate of c_k. Since ρ satisfies the characteristic equation $x^2 + 2c_k x + 1 = 0$, a root ρ' of $x^2 + 2c_k' x + 1 = 0$ is a conjugate of ρ. When $(j, 2q) = 1$, $(\sin(\pi pkj/q))/(\sin(\pi pj/q))$ is a conjugate of c_k. If we let $c_k' = (\sin(\pi pkj/q))/(\sin(\pi pj/q))$, where j is odd and $jp \equiv 1 \pmod q$, then $| c_k' | = |(\sin(\pi k/q))/\sin(\pi/q)| \geqslant 1$. Thus, ρ' is real. Now suppose ρ is a root of unity. Then so is ρ', so $\rho' = \pm 1$. Thus, $\rho = \pm 1$, which contradicts $| c_k | < 1$. Thus, ρ is not a root of unity. ∎

It follows from Lemmas 4 and 5 that $G(\lambda)$ is not discontinuous when $0 < \lambda < 2$, $\lambda \notin C$. We can prove a bit more.

THEOREM 4. *Let $A(\tau)$ be the set of points which are $G(\lambda)$-equivalent to τ. If $0 < \lambda < 2$, $\lambda \notin C$, then for each $\tau \in H$, $A(\tau)$ is dense in H.*

Proof. By Lemma 5, we can find a $W \in G(\lambda)$ such that W has infinite order and W has a fixed point $\tau_1 \in H$. Define $t(\tau) = (\tau - \tau_1)/(\tau - \bar{\tau}_1)$ as before. Assume there is a $\tau \in H$ for which $A(\tau)$ is not dense in H. Then there is an open disk $N \subset H - \{\tau_1\}$ such that $N \cap A(\tau) = \varnothing$. If $\sigma \in K_\alpha \cap A(\tau)$ for some $\alpha \in N$, then N would contain a point in J_σ by Lemma 4, a contradiction. Thus, $K_\alpha \cap A(\tau) = \varnothing$, for each $\alpha \in N$. We can, therefore, find e_1 and e_2 such that

$$\{\sigma \in A(\tau): e_1 < | t(\sigma)| < e_2\} = \varnothing.$$

Let e_3 be the largest number for which $\{\sigma \in A(\tau): e_1 < | t(\sigma)| < e_3\} = \varnothing$. Note $e_3 < 1$, since $| t(S_\lambda{}^m\tau)| \to 1$, as $m \to \infty$. Define β to be the point with the largest real part satisfying $| t(\beta)| = e_3$. Note that β is the right-most point on K_β. The circles K_β and $S_\lambda^{-1}K_{\beta+\lambda}$ intersect at β but they are not tangent because the center of $S_\lambda^{-1}K_{\beta+\lambda}$ is higher than the center of K_β. (The center of K_β is $(x_1, y_1[(2/(1-e_3{}^2)) - 1])$ and the center of $K_{\beta+\lambda}$ is $(x_1, y_1[(2/(1-e_4{}^2)) - 1])$, where $\tau_1 = x_1 + iy_1$ and $e_3 < e_4 = | t(\beta + \lambda)| < 1$). By definition of e_3, there are points of $A(\tau)$ arbitrarily close to K_β. Hence, there are circles $K_\nu(\nu \in A(\tau))$ in any small annulus containing K_β. Lemma 4, thus, shows that β is a cluster point of $A(\tau)$. Choose $\mu \in A(\tau)$ so close to β that K_β and $S_\lambda^{-1}K_{\mu+\lambda}$ intersect but are not tangent. Then there are points of $S_\lambda^{-1}J_{\mu+\lambda}$ in $\{\sigma: e_1 < | t(\sigma)| < e_3\}$, a contradiction. ▌

We conclude with some remarks concerning the distribution of $G(\lambda)$-fixed points in H. A $G(\lambda)$-fixed point is a point in H fixed by some non-identity element of $G(\lambda)$. When $\lambda \geqslant 2$ or $\lambda \in C$, it is clear that $B(\lambda)$ contains no $G(\lambda)$-fixed points. (For suppose $V\tau = \tau$, where $V \in G(\lambda)$, $\tau \in B(\lambda)$. As V is continuous at τ, V maps a neighborhood N of τ into $B(\lambda)$. As no two distinct points of $B(\lambda)$ are $G(\lambda)$-equivalent, V acts as the identity on N. By the identity theorem, $V = I$.)

The following corollary shows that the situation is quite different when $0 < \lambda < 2$, $\lambda \notin C$.

COROLLARY. *If $0 < \lambda < 2$, $\lambda \notin C$, then the set F of $G(\lambda)$-fixed points is dense in H.*

Proof. Let $\tau \in A(i)$, so that $\tau = Vi$ for some $V \in G(\lambda)$. Then $VTV^{-1}\tau = \tau$, so $\tau \in F$. Thus, $A(i) \subset F$ and since $A(i)$ is dense in H by Theorem 4, F is dense in H. ▌

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