MATHEMATICS OF COMPUTATION, VOLUME 33, NUMBER 146
APRIL 1979, PAGES 827–835

The Cyclotomic Numbers of Order Sixteen

By Ronald J. Evans and Jay Roderick Hill

Abstract. A complete table of 408 formulas for cyclotomic numbers of order sixteen is presented. Each number is expressed as a linear combination of parameters of quartic, octic, and bioctic Jacobi sums. Recent applications of these formulas are discussed.

1. Introduction and Notation. Let $p = 16f + 1$ be a prime with a fixed primitive root g. Define the cyclotomic number (i, j) of order sixteen to be the number of integers $n \pmod p$ which satisfy

$$n \equiv g^{16s+i}, \quad 1 + n \equiv g^{16t+j} \pmod p$$

for some choices of s, t in $\{0, 1, 2, \ldots, f - 1\}$. E. Lehmer [10] began the study of these numbers in 1954. A few years later A. L. Whiteman found formulas for the cyclotomic numbers of order sixteen, in terms of parameters of quartic, octic and bioctic Jacobi sums. He gave a table of formulas for $(i, 0)$ in [17]. (The cyclotomic numbers $(i, 0)$ are particularly important, as they have been applied to prove the nonexistence of sixteenth power residue difference sets and modified difference sets; see [17, Section 4], [6].) Most of the formulas for the remaining cyclotomic numbers of order sixteen were not published, and they have apparently been inaccessible. In contrast, complete tables for the cyclotomic numbers of order k are available for $k = 2, 3, 4, 5, 6$ [5]; $k = 7$ [14]; $k = 8$ [11]; $k = 9, 18$ [2]; $k = 10, 12$ [18], [19]; $k = 11$ [13]; $k = 14$ [15]; $k = 20$ [16].

Our objective is to present in Section 3 a complete table of the formulas for the cyclotomic numbers of order sixteen. The computations were performed on the Burroughs 6700 at UCSD, with use of algorithms described in [17, p. 408]. Selected formulas from the tables in Section 3 have been utilized [7] to give elementary resolutions of sign ambiguities in quartic and octic Jacobi and Jacobsthal sums, in certain cases (see Section 2). It is not possible to accomplish these sign resolutions with use of formulas from Whiteman's tables alone. This is what motivated the present work.

We will need the following additional notation. Let $\beta = \exp(2\pi i/16)$, and fix a character $\chi \pmod p$ of order 16 such that $\chi(g) = \beta$. Let m denote the least positive index of 2 with respect to the base g. For any characters λ, $\psi \pmod p$, define

Received January 7, 1978.

AMS (MOS) subject classifications (1970). Primary 10G05, 10A40, 10G04; Secondary 05B10.

Key words and phrases. Cyclotomic numbers of order sixteen, Jacobi sums, Jacobsthal sums, sign ambiguities.

the Jacobi sum

$$J(\lambda, \psi) = \sum_{n \pmod p} \lambda(n)\psi(1-n).$$

In the notation of [17, Section 3], we have

$$J(\chi^4, \chi^4) = -x + 2iy \quad (p = x^2 + 4y^2,\ x \equiv 1 \pmod 4),$$

$$J(\chi^2, \chi^6) = -a + ib\sqrt{2} \quad (p = a^2 + 2b^2,\ a \equiv 1 \pmod 4),$$

$$\begin{aligned}(-1)^f J(\chi, \chi^7) &= c_0 + c_2\sqrt{2} + ic_1\sqrt{2-\sqrt{2}} + ic_3\sqrt{2+\sqrt{2}} \\ &= c_0 + c_2(\beta^2 - \beta^6) + c_1(\beta + \beta^7) + c_3(\beta^3 + \beta^5),\end{aligned}$$

and

$$J(\chi, \chi^2) = \sum_{i=0}^{7} d_i\beta^i.$$

In Section 3, the numbers $256(i, j)$ are expressed as linear combinations of p, 1, x, y, a, b, and the d_i and c_i.

TABLE 1a. f even, $m \equiv 0 \pmod 8$

256 (i , j)	p	1	x	y	a	b	d_0	d_1	d_2	d_3	d_4	d_5	d_6	d_7	c_0	c_1	c_2	c_3
256(0 , 0)	1	-47	-18	0	-48	0	96	0	0	0	0	0	0	0	48	0	0	0
256(0 , 1)	1	-15	2	16	4	24	0	32	16	-16	16	-16	0	0	-8	32	8	0
256(0 , 2)	1	-15	6	16	0	16	-16	0	16	0	16	0	16	0	8	0	32	0
256(0 , 3)	1	-15	2	-16	4	24	0	16	0	32	-16	0	16	16	-8	0	-8	32
256(0 , 4)	1	-15	-2	0	16	0	0	0	0	0	32	0	0	0	0	0	0	0
256(0 , 5)	1	-15	2	16	4	-24	0	16	-16	0	16	32	0	16	-8	0	-8	32
256(0 , 6)	1	-15	6	-16	0	16	-16	0	16	0	-16	0	16	0	8	0	-32	0
256(0 , 7)	1	-15	2	-16	4	-24	0	0	0	-16	-16	-16	-16	32	-8	32	8	0
256(0 , 8)	1	-15	-18	0	-16	0	-32	0	0	0	0	0	0	0	-16	0	0	0
256(0 , 9)	1	-15	2	16	4	24	0	-32	16	16	16	16	0	0	-8	-32	8	0
256(0 , 10)	1	-15	6	16	0	-16	-16	0	-16	0	16	0	-16	0	8	0	-32	0
256(0 , 11)	1	-15	2	-16	4	24	0	-16	0	-32	-16	0	16	-16	-8	0	-8	-32
256(0 , 12)	1	-15	-2	0	16	0	0	0	0	0	-32	0	0	0	0	0	0	0
256(0 , 13)	1	-15	2	16	4	-24	0	-16	-16	0	16	-32	0	-16	-8	0	-8	-32
256(0 , 14)	1	-15	6	-16	0	-16	-16	0	-16	0	-16	0	-16	0	8	0	32	0
256(0 , 15)	1	-15	2	-16	4	-24	0	0	0	16	-16	16	-16	-32	-8	-32	8	0
256(1 , 2)	1	1	2	0	4	0	0	0	-8	0	0	0	8	0	8	0	-8	0
256(1 , 3)	1	1	-6	0	4	0	0	-8	8	8	0	8	-8	24	0	0	-8	-16
256(1 , 4)	1	1	2	0	-4	-8	0	8	0	-8	0	8	16	-8	0	0	8	0
256(1 , 5)	1	1	2	0	-4	-8	0	-8	16	-8	0	8	0	8	0	0	-8	0
256(1 , 6)	1	1	-6	0	4	0	0	-8	-8	24	0	-8	8	-8	0	16	8	0
256(1 , 7)	1	1	2	0	-12	0	0	0	8	-16	0	-16	-8	0	-8	-16	8	16
256(1 , 8)	1	1	2	16	4	8	0	0	-16	0	-16	0	0	0	8	0	-8	0
256(1 , 9)	1	1	2	-16	4	-8	0	0	0	0	16	0	16	0	8	0	-8	0
256(1 , 10)	1	1	2	0	-12	0	0	0	8	16	0	16	-8	0	-8	16	8	-16
256(1 , 11)	1	1	-6	0	4	0	0	8	-8	8	0	-24	8	8	0	-16	8	0
256(1 , 12)	1	1	2	0	-4	8	0	-8	0	-8	0	8	-16	8	0	0	-8	0
256(1 , 13)	1	1	2	0	-4	8	0	8	-16	-8	0	8	0	-8	0	0	8	0
256(1 , 14)	1	1	-6	0	4	0	0	-24	8	-8	0	-8	-8	8	0	0	-8	16
256(2 , 4)	1	1	-2	0	-8	0	-16	0	0	0	0	0	0	0	8	0	0	0
256(2 , 5)	1	1	2	0	-12	0	0	-16	-8	0	0	0	8	-16	-8	16	-8	16
256(2 , 6)	1	1	-2	0	8	16	16	0	-16	0	0	0	-16	0	-8	0	0	0
256(2 , 7)	1	1	-6	0	4	0	0	8	8	-8	0	-8	-8	-24	0	0	-8	16
256(2 , 8)	1	1	6	16	0	0	16	0	0	0	-16	0	0	0	-8	0	0	0
256(2 , 9)	1	1	2	0	4	0	0	0	-8	0	0	0	8	0	8	0	-8	0
256(2 , 10)	1	1	6	-16	0	0	16	0	0	0	16	0	0	0	-8	0	0	0
256(2 , 11)	1	1	-6	0	4	0	0	24	8	8	0	8	-8	-8	0	0	-8	-16
256(2 , 12)	1	1	-2	0	8	-16	16	0	16	0	0	0	16	0	-8	0	0	0
256(2 , 13)	1	1	2	0	-12	0	0	16	-8	0	0	0	8	16	-8	-16	-8	-16
256(3 , 6)	1	1	2	0	4	0	0	0	8	0	0	0	-8	0	8	0	8	0
256(3 , 7)	1	1	2	0	-4	8	0	8	0	8	0	-8	-16	-8	0	0	-8	0
256(3 , 8)	1	1	2	-16	4	8	0	0	0	0	16	0	-16	0	8	0	8	0
256(3 , 9)	1	1	-6	0	4	0	0	-8	-8	-8	0	24	8	-8	0	16	8	0
256(3 , 10)	1	1	-6	0	4	0	0	8	-8	-24	0	8	8	8	0	-16	8	0
256(3 , 11)	1	1	2	16	4	-8	0	0	16	0	-16	0	0	0	8	0	8	0
256(3 , 12)	1	1	2	0	-4	-8	0	8	16	8	0	-8	0	-8	0	0	-8	0
256(4 , 8)	1	1	-2	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
256(4 , 9)	1	1	2	0	-4	-8	0	-8	0	8	0	-8	16	8	0	0	8	0
256(4 , 10)	1	1	-2	0	-8	0	-16	0	0	0	0	0	0	0	8	0	0	0
256(4 , 11)	1	1	2	0	-4	8	0	-8	-16	8	0	-8	0	8	0	0	8	0
256(5 , 10)	1	1	2	0	4	0	0	0	8	0	0	0	-8	0	8	0	8	0

TABLE 1b. f odd, $m \equiv 0 \pmod 8$

256 (i , j)	P	l	x	y	a	b	d_0	d_1	d_2	d_3	d_4	d_5	d_6	d_7	c_0	c_1	c_2	c_3
256(0 , 0)	1	-31	-18	0	-16	0	0	0	0	0	0	0	0	0	0	0	0	0
256(0 , 1)	1	1	2	16	4	24	0	0	-16	16	16	-16	0	0	8	0	8	0
256(0 , 2)	1	1	6	16	0	-16	-16	0	-16	0	-16	0	-16	0	-8	0	0	0
256(0 , 3)	1	1	2	-16	4	24	0	-16	0	0	-16	0	-16	16	8	0	-8	0
256(0 , 4)	1	1	-2	0	16	0	32	0	0	0	32	0	0	0	-16	0	0	0
256(0 , 5)	1	1	2	16	4	-24	0	16	16	0	16	0	0	-16	8	0	-8	0
256(0 , 6)	1	1	6	-16	0	-16	-16	0	-16	0	16	0	-16	0	-8	0	0	0
256(0 , 7)	1	1	2	-16	4	-24	0	0	0	-16	-16	16	16	0	8	0	8	0
256(0 , 8)	1	1	-18	0	-48	0	0	0	0	0	0	0	0	0	0	0	0	0
256(0 , 9)	1	1	2	16	4	24	0	0	-16	-16	16	16	0	0	8	0	8	0
256(0 , 10)	1	1	6	16	0	16	-16	0	16	0	-16	0	16	0	-8	0	0	0
256(0 , 11)	1	1	2	-16	4	24	0	16	0	0	-16	0	-16	-16	8	0	-8	0
256(0 , 12)	1	1	-2	0	16	0	32	0	0	0	-32	0	0	0	-16	0	0	0
256(0 , 13)	1	1	2	16	4	-24	0	-16	16	0	16	0	0	16	8	0	-8	0
256(0 , 14)	1	1	6	-16	0	16	-16	0	16	0	16	0	16	0	-8	0	0	0
256(0 , 15)	1	1	2	-16	4	-24	0	0	0	16	-16	-16	16	0	8	0	8	0
256(1 , 0)	1	-15	2	16	4	8	0	0	16	0	-16	0	0	0	-8	0	-8	0
256(1 , 1)	1	-15	2	-16	4	-8	0	0	0	0	16	0	-16	0	-8	0	-8	0
256(1 , 2)	1	1	2	0	-12	0	0	-16	8	0	0	0	-8	-16	-8	-16	-8	-16
256(1 , 3)	1	1	-6	0	4	0	0	-8	-8	-8	0	-8	8	-8	0	0	-8	-16
256(1 , 4)	1	1	2	0	-4	8	0	24	0	-8	0	8	16	8	0	-16	-8	-16
256(1 , 5)	1	1	2	0	-4	8	0	8	16	24	0	8	0	-8	0	16	8	-16
256(1 , 6)	1	1	-6	0	4	0	0	-8	8	8	0	8	-8	-8	0	-16	8	0
256(1 , 7)	1	1	2	0	4	0	0	-16	-8	16	0	-16	8	16	8	0	8	0
256(1 , 11)	1	1	-6	0	4	0	0	8	8	-8	0	-8	-8	8	0	16	8	0
256(1 , 12)	1	1	2	0	-4	-8	0	8	0	-8	0	-24	-16	-8	0	-16	8	16
256(1 , 13)	1	1	2	0	-4	-8	0	-8	-16	-8	0	8	0	-24	0	16	-8	16
256(1 , 14)	1	1	-6	0	4	0	0	8	-8	8	0	8	8	8	0	0	-8	16
256(1 , 15)	1	1	2	0	-12	0	0	16	8	0	0	0	-8	16	-8	16	-8	16
256(2 , 0)	1	-15	6	16	0	0	16	0	0	0	16	0	0	0	8	0	0	0
256(2 , 1)	1	1	2	0	4	0	0	16	-8	-16	0	16	8	-16	8	0	8	0
256(2 , 2)	1	-15	6	-16	0	0	16	0	0	0	-16	0	0	0	8	0	0	0
256(2 , 3)	1	1	-6	0	4	0	0	8	8	-8	0	-8	-8	8	0	16	8	0
256(2 , 4)	1	1	-2	0	8	-16	-16	0	16	0	0	0	16	0	-8	0	0	0
256(2 , 5)	1	1	2	0	-12	0	0	0	-8	16	0	16	8	0	-8	-16	8	16
256(2 , 6)	1	1	-2	0	-8	0	16	0	0	0	0	0	0	0	8	0	-32	0
256(2 , 13)	1	1	2	0	-12	0	0	0	-8	-16	0	-16	8	0	-8	16	8	-16
256(2 , 14)	1	1	-2	0	8	16	-16	0	-16	0	0	0	-16	0	-8	0	0	0
256(2 , 15)	1	1	-6	0	4	0	0	-8	8	8	0	8	-8	-8	0	-16	8	0
256(3 , 0)	1	-15	2	-16	4	8	0	0	0	0	16	0	16	0	-8	0	8	0
256(3 , 1)	1	1	-6	0	4	0	0	8	-8	8	0	8	8	8	0	0	-8	16
256(3 , 2)	1	1	-6	0	4	0	0	-8	-8	-8	0	-8	8	-8	0	0	-8	-16
256(3 , 3)	1	-15	2	16	4	-8	0	0	-16	0	-16	0	0	0	-8	0	8	0
256(3 , 4)	1	1	2	0	-4	-8	0	8	-16	8	0	-8	0	24	0	-16	-8	-16
256(3 , 5)	1	1	2	0	4	0	0	-16	8	-16	0	16	-8	16	8	0	-8	0
256(3 , 15)	1	1	2	0	-4	8	0	-24	0	8	0	-8	16	-8	0	16	-8	16
256(4 , 0)	1	-15	-2	0	0	0	-32	0	0	0	0	0	0	0	16	0	0	0
256(4 , 1)	1	1	2	0	-4	-8	0	-8	0	8	0	24	-16	8	0	16	8	-16
256(4 , 2)	1	1	-2	0	-8	0	16	0	0	0	0	0	0	0	8	0	32	0
256(4 , 3)	1	1	2	0	-4	8	0	-8	16	-24	0	-8	0	8	0	-16	8	16
256(5 , 2)	1	1	2	0	4	0	0	16	8	16	0	-16	-8	-16	8	0	-8	0

2. **Applications.** It is known that $c_0 \equiv -1 \pmod 8$ and that $c_2 \equiv m \pmod 4$; see [8, p. 338], [4, Theorems 3.5 and 3.6]. With the use of the tables in Section 3, the values of c_0 and c_2 have been characterized [7] modulo 16 and 8, respectively. For example, in the case that $2 | f$ and $8 | m$ (so that $4 | b$ by [1] or [3, Theorem 3.15]), we have

$$c_0 \equiv -1 \pmod{16} \quad \text{and} \quad c_2 \equiv 0 \pmod 8, \quad \text{if } 8 | b,$$

$$c_0 \equiv 7 \pmod{16} \quad \text{and} \quad c_2 \equiv 4 \pmod 8, \quad \text{otherwise.}$$

The tables in Section 3 have also been applied [7] to give an elementary proof of the important relation

$$y \equiv 2b + m \pmod{16}. \tag{1}$$

Hasse [9, p. 232] deduced (1) using deep methods from class field theory. Whiteman [17, Section 5] gave an elementary proof of (1) in the case $8 | m$ and, thereby, supplied a relatively simple demonstration of the Cunningham-Aigner criterion for the sixteenth power residue character of 2. In the case $8 \nmid m$, (1) leads to reso-

TABLE 2a. f even, $m \equiv \pm 2 \pmod 8$

256 (i , j)	p	1	x	y	a	b	d_0	d_1	d_2	d_3	d_4	d_5	d_6	d_7	c_0	c_1	c_2	c_3
256(0 , 0)	1	-47	6	0	0	0	48	0	0	0	48	0	0	0	24	0	0	0
256(0 , 1)	1	-15	2	0	4	-8	-16	16	16	0	0	0	0	-16	-8	16	8	16
256(0 , 2)	1	-15	-2	-16	-16	16	0	0	-16	0	0	0	16	0	16	0	0	0
256(0 , 3)	1	-15	2	0	4	8	16	0	0	16	0	16	16	32	-8	16	8	16
256(0 , 4)	1	-15	-10	0	16	0	-16	0	0	0	48	0	0	0	-8	0	0	0
256(0 , 5)	1	-15	2	0	4	8	-16	0	-16	-16	0	16	0	0	-8	-16	-8	16
256(0 , 6)	1	-15	-2	16	0	-16	0	0	-16	0	-32	0	16	0	0	0	-32	0
256(0 , 7)	1	-15	2	0	4	-8	16	16	0	-32	0	0	-16	16	-8	16	-8	-16
256(0 , 8)	1	-15	6	0	0	0	-16	0	0	0	-16	0	0	0	-8	0	0	0
256(0 , 9)	1	-15	2	0	4	-8	-16	-16	16	0	0	0	0	16	-8	-16	8	-16
256(0 , 10)	1	-15	-2	-16	-16	-16	0	0	16	0	0	0	-16	0	16	0	0	0
256(0 , 11)	1	-15	2	0	4	8	16	0	0	-16	0	-16	16	-32	-8	-16	8	-16
256(0 , 12)	1	-15	-10	0	-16	0	-16	0	0	0	-16	0	0	0	24	0	0	0
256(0 , 13)	1	-15	2	0	4	8	-16	0	-16	16	0	-16	0	0	-8	16	-8	-16
256(0 , 14)	1	-15	-2	16	0	16	0	0	16	0	-32	0	-16	0	0	0	32	0
256(0 , 15)	1	-15	2	0	4	-8	16	-16	0	32	0	0	-16	-16	-8	-16	-8	16
256(1 , 2)	1	1	-6	0	4	0	0	16	-8	-16	0	0	-8	0	0	16	-8	0
256(1 , 3)	1	1	2	0	-12	-16	0	-8	8	-8	0	8	8	8	-8	-16	-8	0
256(1 , 4)	1	1	2	16	-4	-8	0	8	0	8	0	24	16	-8	0	0	-8	-16
256(1 , 5)	1	1	2	-16	-4	8	0	-8	16	8	0	-8	0	8	0	0	-8	-16
256(1 , 6)	1	1	2	0	-12	16	0	-8	-8	8	0	-8	-8	8	-8	0	8	-16
256(1 , 7)	1	1	-6	0	4	0	0	0	8	-16	0	0	8	-16	0	-16	8	0
256(1 , 8)	1	1	2	0	4	8	16	0	-16	0	0	0	0	0	8	0	-8	0
256(1 , 9)	1	1	2	0	4	8	-16	0	0	0	0	0	16	0	8	0	8	0
256(1 , 10)	1	1	-6	0	4	0	0	0	8	16	0	0	8	16	0	16	8	0
256(1 , 11)	1	1	2	0	4	-16	0	24	-8	8	0	-8	-8	8	8	-16	8	0
256(1 , 12)	1	1	2	16	-4	8	0	-24	0	-8	0	8	-16	-8	0	16	8	0
256(1 , 13)	1	1	2	-16	-4	-8	0	-8	-16	-8	0	8	0	8	0	-16	8	0
256(1 , 14)	1	1	2	0	4	16	0	-8	8	-8	0	-24	8	8	8	0	-8	16
256(2 , 4)	1	1	6	0	8	-16	-16	0	0	0	0	0	0	0	0	0	16	0
256(2 , 5)	1	1	-6	0	4	0	0	0	-8	16	0	0	-8	-16	0	0	-8	16
256(2 , 6)	1	1	6	0	8	0	16	0	16	0	0	0	-16	0	0	0	-16	0
256(2 , 7)	1	1	2	0	-12	-16	0	8	8	8	0	-8	8	-8	-8	16	-8	0
256(2 , 8)	1	1	-2	-16	0	0	0	0	0	0	0	0	0	0	-16	0	0	0
256(2 , 9)	1	1	-6	0	4	0	0	-16	-8	16	0	0	-8	0	0	-16	-8	0
256(2 , 10)	1	1	-2	16	-16	0	0	0	0	0	32	0	0	0	0	0	0	0
256(2 , 11)	1	1	2	0	4	16	0	8	8	8	0	24	8	-8	8	0	-8	-16
256(2 , 12)	1	1	6	0	8	0	16	0	-16	0	0	0	16	0	0	0	16	0
256(2 , 13)	1	1	-6	0	4	0	0	0	-8	-16	0	0	-8	16	0	0	-8	-16
256(3 , 6)	1	1	-6	0	4	0	0	0	8	0	0	-16	8	-16	0	0	8	-16
256(3 , 7)	1	1	2	16	-4	8	0	24	0	8	0	-8	-16	8	0	-16	8	0
256(3 , 8)	1	1	2	0	4	-8	-16	0	0	0	0	0	-16	0	8	0	-8	0
256(3 , 9)	1	1	2	0	4	-16	0	-24	-8	-8	0	8	-8	-8	8	16	8	0
256(3 , 10)	1	1	2	0	-12	16	0	8	-8	-8	0	8	-8	-8	-8	0	8	16
256(3 , 11)	1	1	2	0	4	-8	16	0	16	0	0	0	0	0	8	0	8	0
256(3 , 12)	1	1	2	-16	-4	8	0	8	16	-8	0	8	0	-8	0	0	-8	16
256(4 , 8)	1	1	-10	0	0	0	16	0	0	0	-16	0	0	0	-8	0	0	0
256(4 , 9)	1	1	2	16	-4	-8	0	-8	0	-8	0	-24	16	8	0	0	-8	16
256(4 , 10)	1	1	6	0	8	16	-16	0	0	0	0	0	0	0	0	0	-16	0
256(4 , 11)	1	1	2	-16	-4	-8	0	8	-16	8	0	-8	0	-8	0	16	8	0
256(5 , 10)	1	1	-6	0	4	0	0	0	8	0	0	16	8	16	0	0	8	16
256 (- i , -j)	p	1	x	-y	a	-b	d_0	$-d_7$	$-d_6$	$-d_5$	$-d_4$	$-d_3$	$-d_2$	$-d_1$	c_0	$-c_1$	c_2	$-c_3$

lutions of sign ambiguities in quartic and octic Jacobi and Jacobsthal sums. For example, in the case that 2 is a quartic but not octic residue (mod p), i.e., $m \equiv \pm 4 \pmod{16}$, the sign of y is determined by (1), because in this case $4 | b$ by [1] or [3, Theorem 3.15]. This sign determination extends the results of E. Lehmer for quartic sums [11], [12]. Sign ambiguities in octic sums have been resolved only in the case that 2 is a quartic nonresidue (mod p), i.e., $m \equiv \pm 2 \pmod 8$. In this case, the sign of b is determined by (1), since here $y \equiv -m \pmod 8$ by [11, p. 108].

Consider now primes $p \equiv 1 \pmod{32}$. For such primes, Hasse [9, p. 233] proved that

$$y + 2b - 4(c_1 + c_3) \equiv m \pmod{32}. \tag{2}$$

In the case that $y \equiv m \equiv 0 \pmod{16}$, Hasse showed that (2) yields a simple unam-

TABLE 2b. f odd, $m \equiv \pm 2 \pmod 8$

256 (i , j)	p	1	x	y	a	b	d_0	d_1	d_2	d_3	d_4	d_5	d_6	d_7	c_0	c_1	c_2	c_3
256(0 , 0)	1	-31	6	0	0	0	16	0	0	0	-16	0	0	0	8	0	0	0
256(0 , 1)	1	1	2	0	4	-8	-16	16	-16	0	0	-32	0	16	8	16	8	-16
256(0 , 2)	1	1	-2	-16	-16	-16	0	0	16	0	-32	0	-16	0	0	0	32	0
256(0 , 3)	1	1	2	0	4	8	16	0	0	16	0	-16	-16	0	8	-16	8	16
256(0 , 4)	1	1	-10	0	-16	0	16	0	0	0	48	0	0	0	8	0	0	0
256(0 , 5)	1	1	2	0	4	8	-16	32	16	16	0	16	0	0	8	16	-8	16
256(0 , 6)	1	1	-2	16	0	16	0	0	16	0	0	0	-16	0	-16	0	0	0
256(0 , 7)	1	1	2	0	4	-8	16	-16	0	0	0	0	16	16	8	16	-8	16
256(0 , 8)	1	1	6	0	0	0	-48	0	0	0	48	0	0	0	-24	0	0	0
256(0 , 9)	1	1	2	0	4	-8	-16	-16	-16	0	0	32	0	-16	8	-16	8	16
256(0 , 10)	1	1	-2	-16	-16	16	0	0	-16	0	-32	0	16	0	0	0	-32	0
256(0 , 11)	1	1	2	0	4	8	16	0	0	-16	0	16	-16	0	8	16	8	-16
256(0 , 12)	1	1	-10	0	16	0	16	0	0	0	-16	0	0	0	-24	0	0	0
256(0 , 13)	1	1	2	0	4	8	-16	-32	16	-16	0	-16	0	0	8	-16	-8	-16
256(0 , 14)	1	1	-2	16	0	-16	0	0	-16	0	0	0	16	0	-16	0	0	0
256(0 , 15)	1	1	2	0	4	-8	16	16	0	0	0	0	16	-16	8	-16	-8	-16
256(1 , 0)	1	-15	2	0	4	8	16	0	16	0	0	0	0	0	-8	0	-8	0
256(1 , 1)	1	-15	2	0	4	8	-16	0	0	0	0	0	-16	0	-8	0	8	0
256(1 , 2)	1	1	-6	0	4	0	0	-16	8	0	0	-16	8	0	0	-16	-8	0
256(1 , 3)	1	1	2	0	4	-16	0	8	-8	-8	0	8	-8	-8	8	0	-8	-16
256(1 , 4)	1	1	2	16	-4	8	0	8	0	-8	0	8	16	-8	0	0	8	-16
256(1 , 5)	1	1	2	-16	-4	-8	0	-8	16	24	0	8	0	8	0	0	8	-16
256(1 , 6)	1	1	2	0	4	16	0	8	8	8	0	-8	8	-8	8	-16	8	0
256(1 , 7)	1	1	-6	0	4	0	0	0	-8	0	0	-16	-8	16	0	16	8	0
256(1 , 11)	1	1	2	0	-12	-16	0	8	8	-24	0	-8	8	-8	-8	0	8	16
256(1 , 12)	1	1	2	16	-4	-8	0	8	0	8	0	-8	-16	-8	0	-16	-8	0
256(1 , 13)	1	1	2	-16	-4	8	0	-8	-16	8	0	-8	0	-24	0	16	-8	0
256(1 , 14)	1	1	2	0	-12	16	0	8	-8	-8	0	8	-8	24	-8	-16	-8	0
256(1 , 15)	1	1	-6	0	4	0	0	16	8	0	0	16	8	0	0	16	-8	0
256(2 , 0)	1	-15	-2	-16	0	0	0	0	0	0	32	0	0	0	0	0	0	0
256(2 , 1)	1	1	-6	0	4	0	0	0	-8	0	0	16	-8	-16	0	-16	8	0
256(2 , 2)	1	-15	-2	16	-16	0	0	0	0	0	0	0	0	0	16	0	0	0
256(2 , 3)	1	1	2	0	4	16	0	-8	8	-8	0	8	8	8	8	16	8	0
256(2 , 4)	1	1	6	0	8	0	-16	0	-16	0	0	0	16	0	0	0	16	0
256(2 , 5)	1	1	-6	0	4	0	0	-16	-8	0	0	16	-8	0	0	0	8	16
256(2 , 6)	1	1	6	0	8	-16	16	0	0	0	0	0	0	0	0	0	-16	0
256(2 , 13)	1	1	-6	0	4	0	0	16	-8	0	0	-16	-8	0	0	0	8	-16
256(2 , 14)	1	1	6	0	8	0	-16	0	16	0	0	0	-16	0	0	0	-16	0
256(2 , 15)	1	1	2	0	-12	-16	0	-8	8	24	0	8	8	8	-8	0	8	-16
256(3 , 0)	1	-15	2	0	4	-8	-16	0	0	0	0	0	16	0	-8	0	-8	0
256(3 , 1)	1	1	2	0	4	-16	0	-8	-8	8	0	-8	-8	8	8	0	-8	16
256(3 , 2)	1	1	2	0	-12	16	0	-8	-8	8	0	-8	-8	-24	-8	16	-8	0
256(3 , 3)	1	-15	2	0	4	-8	16	0	-16	0	0	0	0	0	-8	0	8	0
256(3 , 4)	1	1	2	-16	-4	8	0	8	-16	-8	0	8	0	24	0	-16	-8	0
256(3 , 5)	1	1	-6	0	4	0	0	-16	8	-16	0	0	8	0	0	0	-8	-16
256(3 , 15)	1	1	2	16	-4	8	0	-8	0	8	0	-8	16	8	0	0	8	16
256(4 , 0)	1	-15	-10	0	0	0	-16	0	0	0	-16	0	0	0	8	0	0	0
256(4 , 1)	1	1	2	16	-4	-8	0	-8	0	-8	0	8	-16	8	0	16	-8	0
256(4 , 2)	1	1	6	0	8	16	16	0	0	0	0	0	0	0	0	0	16	0
256(4 , 3)	1	1	2	-16	-4	-8	0	8	16	-24	0	-8	0	-8	0	0	8	16
256(5 , 2)	1	1	-6	0	4	0	0	16	8	16	0	0	8	0	0	0	-8	16
256 (-i , -j)	p	1	x	-y	a	-b	d_0	$-d_7$	$-d_6$	$-d_5$	$-d_4$	$-d_3$	$-d_2$	$-d_1$	c_0	$-c_1$	c_2	$-c_3$

biguous criterion for the 32nd power residue character of 2. In the case that $4 \nmid m$, (2) leads [7] to a resolution of signs for bioctic Jacobi and Jacobsthal sums. (The latter sums are evaluated up to sign in [4, Theorem 3.9].) No elementary proof of (2) appears to be known, but such a proof could undoubtedly be found once a table of cyclotomic numbers or order 32 has been constructed. Note that (1) and (2) are special cases of J. B. Muskat's equation (9) on p. 146 of *Computers in Number Theory*, Academic Press, New York, 1971.

3. Tables of (i, j). Of the 256 possible formulas for (i, j) corresponding to a given prime p, at most 51 are distinct. These formulas are given in Tables 1a–3a for f even, and in Tables 1b–3b for f odd. Tables 1a, 1b (resp., 3a, 3b) treat the primes p for which $m \equiv 0 \pmod 8$ (resp., $m \equiv 4 \pmod 8$)). Tables 2a, 2b treat the

TABLE 3a. f even, $m \equiv 4 \pmod 8$

256(i , j)	p	l	x	y	a	b	d_0	d_1	d_2	d_3	d_4	d_5	d_6	d_7	c_0	c_1	c_2	c_3
256(0 , 0)	1	-47	-18	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
256(0 , 1)	1	-15	2	16	4	8	0	0	16	16	-16	-16	0	0	-8	0	-8	0
256(0 , 2)	1	-15	6	16	0	-16	16	0	-16	0	16	0	-16	0	8	0	0	0
256(0 , 3)	1	-15	2	-16	4	8	0	-16	0	0	16	0	16	16	-8	0	8	0
256(0 , 4)	1	-15	-2	0	0	0	-32	0	0	0	32	0	0	0	16	0	0	0
256(0 , 5)	1	-15	2	16	4	-8	0	16	-16	0	-16	0	0	-16	-8	0	8	0
256(0 , 6)	1	-15	6	-16	0	-16	16	0	-16	0	-16	0	-16	0	8	0	0	0
256(0 , 7)	1	-15	2	-16	4	-8	0	0	0	-16	16	16	-16	0	-8	0	-8	0
256(0 , 8)	1	-15	-18	0	-32	0	0	0	0	0	0	0	0	0	0	0	0	0
256(0 , 9)	1	-15	2	16	4	8	0	0	16	-16	-16	16	0	0	-8	0	-8	0
256(0 , 10)	1	-15	6	16	0	16	16	0	16	0	16	0	16	0	8	0	0	0
256(0 , 11)	1	-15	2	-16	4	8	0	16	0	0	16	0	16	-16	-8	0	8	0
256(0 , 12)	1	-15	-2	0	0	0	-32	0	0	0	-32	0	0	0	16	0	0	0
256(0 , 13)	1	-15	2	16	4	-8	0	-16	-16	0	-16	0	0	16	-8	0	8	0
256(0 , 14)	1	-15	6	-16	0	16	16	0	16	0	-16	0	16	0	8	0	0	0
256(0 , 15)	1	-15	2	-16	4	-8	0	0	0	16	16	-16	-16	0	-8	0	-8	0
256(1 , 2)	1	1	2	0	-12	0	0	16	8	-16	0	16	-8	-16	-8	0	-8	0
256(1 , 3)	1	1	-6	0	4	0	0	-8	-8	8	0	8	8	-8	0	-16	-8	0
256(1 , 4)	1	1	2	0	-4	8	0	-8	0	8	0	24	16	8	0	16	-8	-16
256(1 , 5)	1	1	2	0	-4	8	0	8	16	8	0	-8	0	24	0	-16	8	-16
256(1 , 6)	1	1	-6	0	4	0	0	-8	8	-8	0	-8	-8	-8	0	0	8	-16
256(1 , 7)	1	1	2	0	4	0	0	-16	-8	0	0	0	8	-16	8	-16	8	-16
256(1 , 8)	1	1	2	16	4	24	0	0	-16	0	16	0	0	0	8	0	8	0
256(1 , 9)	1	1	2	-16	4	-24	0	0	0	0	-16	0	16	0	8	0	8	0
256(1 , 10)	1	1	2	0	4	0	0	16	-8	0	0	0	8	16	8	16	8	16
256(1 , 11)	1	1	-6	0	4	0	0	8	8	8	0	8	-8	8	0	0	8	16
256(1 , 12)	1	1	2	0	-4	-8	0	-24	0	8	0	-8	-16	-8	0	16	8	16
256(1 , 13)	1	1	2	0	-4	-8	0	-8	-16	-24	0	-8	0	8	0	-16	-8	16
256(1 , 14)	1	1	-6	0	4	0	0	8	-8	-8	0	-8	8	8	0	16	-8	0
256(2 , 4)	1	1	-2	0	8	0	-16	0	0	0	0	0	0	0	-8	0	32	0
256(2 , 5)	1	1	2	0	4	0	0	0	8	16	0	16	-8	0	8	-16	-8	16
256(2 , 6)	1	1	-2	0	-8	-16	16	0	16	0	0	0	16	0	8	0	0	0
256(2 , 7)	1	1	-6	0	4	0	0	8	-8	-8	0	-8	8	8	0	16	-8	0
256(2 , 8)	1	1	6	16	0	0	-16	0	0	0	-16	0	0	0	-8	0	0	0
256(2 , 9)	1	1	2	0	-12	0	0	-16	8	16	0	-16	-8	16	-8	0	-8	0
256(2 , 10)	1	1	6	-16	0	0	-16	0	0	0	16	0	0	0	-8	0	0	0
256(2 , 11)	1	1	-6	0	4	0	0	-8	-8	8	0	8	8	-8	0	-16	-8	0
256(2 , 12)	1	1	-2	0	-8	16	16	0	-16	0	0	0	-16	0	8	0	0	0
256(2 , 13)	1	1	2	0	4	0	0	0	8	-16	0	-16	-8	0	8	16	-8	-16
256(3 , 6)	1	1	2	0	-12	0	0	16	-8	16	0	-16	8	-16	-8	0	8	0
256(3 , 7)	1	1	2	0	-4	-8	0	24	0	-8	0	8	-16	8	0	-16	8	-16
256(3 , 8)	1	1	2	-16	4	24	0	0	0	0	-16	0	-16	0	8	0	-8	0
256(3 , 9)	1	1	-6	0	4	0	0	-8	8	-8	0	-8	-8	-8	0	0	8	-16
256(3 , 10)	1	1	-6	0	4	0	0	8	8	8	0	8	-8	8	0	0	8	16
256(3 , 11)	1	1	2	16	4	-24	0	0	16	0	16	0	0	0	8	0	-8	0
256(3 , 12)	1	1	2	0	-4	8	0	-8	16	-8	0	8	0	-24	0	16	8	16
256(4 , 8)	1	1	-2	0	16	0	32	0	0	0	0	0	0	0	-16	0	0	0
256(4 , 9)	1	1	2	0	-4	8	0	8	0	-8	0	-24	16	-8	0	-16	-8	16
256(4 , 10)	1	1	-2	0	8	0	-16	0	0	0	0	0	0	0	-8	0	-32	0
256(4 , 11)	1	1	2	0	-4	-8	0	8	-16	24	0	8	0	-8	0	16	-8	-16
256(5 , 10)	1	1	2	0	-12	0	0	-16	-8	-16	0	16	8	16	-8	0	8	0

cases $m \equiv \pm 2 \pmod 8$, where the upper headings are used for $m \equiv 2 \pmod 8$ and the lower headings are used for $m \equiv -2 \pmod 8$. For example, from Table 2a, we find that

$$256(0, 2) = p - 15 - 2x - 16y - 16a + 16b - 16d_2 + 16d_6 + 16c_0,$$

when $m \equiv 2 \pmod 8$, whereas

$$256(0, 2) = 256(0, -14)$$

$$= p - 15 - 2x - 16y - 16b + 16d_2 - 16d_6 + 32d_4 + 32c_2,$$

when $m \equiv -2 \pmod 8$. To obtain a formula for a number (i, j) not listed in a given one of Tables 1a–3a or 1b–3b, one would consult Table 4a or 4b according as f is even or odd. For example, from Table 4b, one finds that $(11, 4) = (1, 13)$.

TABLE 3b. f odd, $m \equiv 4 \pmod 8$

256(i , j)	p	l	x	y	a	b	d_0	d_1	d_2	d_3	d_4	d_5	d_6	d_7	c_0	c_1	c_2	c_3
256(0 , 0)	1	-31	-18	0	-32	0	32	0	0	0	0	0	0	0	16	0	0	0
256(0 , 1)	1	1	2	16	4	8	0	32	-16	-16	-16	-16	0	0	8	32	-8	0
256(0 , 2)	1	1	6	16	0	16	16	0	16	0	-16	0	16	0	-8	0	32	0
256(0 , 3)	1	1	2	-16	4	8	0	16	0	32	16	0	-16	16	8	0	8	32
256(0 , 4)	1	1	-2	0	0	0	0	0	0	0	32	0	0	0	0	0	0	0
256(0 , 5)	1	1	2	16	4	-8	0	16	16	0	-16	32	0	16	8	0	8	32
256(0 , 6)	1	1	6	-16	0	16	16	0	16	0	16	0	16	0	-8	0	-32	0
256(0 , 7)	1	1	2	-16	4	-8	0	0	0	-16	16	-16	16	32	8	32	-8	0
256(0 , 8)	1	1	-18	0	0	0	-96	0	0	0	0	0	0	0	-48	0	0	0
256(0 , 9)	1	1	2	16	4	8	0	-32	-16	16	-16	16	0	0	8	-32	-8	0
256(0 , 10)	1	1	6	16	0	-16	16	0	-16	0	-16	0	-16	0	-8	0	-32	0
256(0 , 11)	1	1	2	-16	4	8	0	-16	0	-32	16	0	-16	-16	8	0	8	-32
256(0 , 12)	1	1	-2	0	0	0	0	0	0	0	-32	0	0	0	0	0	0	0
256(0 , 13)	1	1	2	16	4	-8	0	-16	16	0	-16	-32	0	-16	8	0	8	-32
256(0 , 14)	1	1	6	-16	0	-16	16	0	-16	0	16	0	-16	0	-8	0	32	0
256(0 , 15)	1	1	2	-16	4	-8	0	0	0	16	16	16	16	-32	8	-32	-8	0
256(1 , 0)	1	-15	2	16	4	24	0	0	16	0	16	0	0	0	-8	0	8	0
256(1 , 1)	1	-15	2	-16	4	-24	0	0	0	0	-16	0	-16	0	-8	0	8	0
256(1 , 2)	1	1	2	0	4	0	0	0	-8	-16	0	-16	8	0	8	-16	-8	16
256(1 , 3)	1	1	-6	0	4	0	0	-8	8	-8	0	24	-8	-8	0	16	-8	0
256(1 , 4)	1	1	2	0	-4	-8	0	8	0	8	0	-8	16	-8	0	0	8	0
256(1 , 5)	1	1	2	0	-4	-8	0	-8	16	8	0	-8	0	8	0	0	-8	0
256(1 , 6)	1	1	-6	0	4	0	0	24	-8	8	0	8	8	-8	0	0	8	-16
256(1 , 7)	1	1	2	0	-12	0	0	0	8	0	0	0	-8	0	-8	0	8	0
256(1 , 11)	1	1	-6	0	4	0	0	8	-8	-8	0	-8	8	-24	0	0	8	16
256(1 , 12)	1	1	2	0	-4	8	0	-8	0	8	0	-8	-16	8	0	0	-8	0
256(1 , 13)	1	1	2	0	-4	8	0	8	-16	8	0	-8	0	-8	0	0	8	0
256(1 , 14)	1	1	-6	0	4	0	0	8	8	-24	0	8	-8	8	0	-16	-8	0
256(1 , 15)	1	1	2	0	4	0	0	0	-8	16	0	16	8	0	8	16	-8	-16
256(2 , 0)	1	-15	6	16	0	0	-16.	0	0	0	16	0	0	0	8	0	0	0
256(2 , 1)	1	1	2	0	-12	0	0	0	8	0	0	0	-8	0	-8	0	8	0
256(2 , 2)	1	-15	6	-16	0	0	-16	0	0	0	-16	0	0	0	8	0	0	0
256(2 , 3)	1	1	-6	0	4	0	0	-24	-8	-8	0	-8	8	8	0	0	8	16
256(2 , 4)	1	1	-2	0	-8	16	-16	0	-16	0	0	0	-16	0	8	0	0	0
256(2 , 5)	1	1	2	0	4	0	0	-16	8	0	0	0	-8	-16	8	16	8	16
256(2 , 6)	1	1	-2	0	8	0	16	0	0	0	0	0	0	0	-8	0	0	0
256(2 , 13)	1	1	2	0	4	0	0	16	8	0	0	0	-8	16	8	-16	8	-16
256(2 , 14)	1	1	-2	0	-8	-16	-16	0	16	0	0	0	16	0	8	0	0	0
256(2 , 15)	1	1	-6	0	4	0	0	-8	-8	8	0	8	8	24	0	0	8	-16
256(3 , 0)	1	-15	2	-16	4	24	0	0	0	0	-16	0	16	0	-8	0	-8	0
256(3 , 1)	1	1	-6	0	4	0	0	8	8	8	0	-24	-8	8	0	-16	-8	0
256(3 , 2)	1	1	-6	0	4	0	0	-8	8	24	0	-8	-8	-8	0	16	-8	0
256(3 , 3)	1	-15	2	16	4	-24	0	0	-16	0	16	0	0	0	-8	0	-8	0
256(3 , 4)	1	1	2	0	-4	8	0	-8	-16	-8	0	8	0	8	0	0	8	0
256(3 , 5)	1	1	2	0	-12	0	0	0	-8	0	0	0	8	0	-8	0	-8	0
256(3 , 15)	1	1	2	0	-4	-8	0	-8	0	-8	0	8	16	8	0	0	8	0
256(4 , 0)	1	-15	-2	0	16	0	0	0	0	0	0	0	0	0	0	0	0	0
256(4 , 1)	1	1	2	0	-4	8	0	8	0	-8	0	8	-16	-8	0	0	-8	0
256(4 , 2)	1	1	-2	0	8	0	16	0	0	0	0	0	0	0	-8	0	0	0
256(4 , 3)	1	1	2	0	-4	-8	0	8	16	-8	0	8	0	-8	0	0	-8	0
256(5 , 2)	1	1	2	0	-12	0	0	0	-8	0	0	0	8	0	-8	0	-8	0

TABLE 4a. f even

i \ j	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15
0	0,0	0,1	0,2	0,3	0,4	0,5	0,6	0,7	0,8	0,9	0,10	0,11	0,12	0,13	0,14	0,15
1	0,1	0,15	1,2	1,3	1,4	1,5	1,6	1,7	1,8	1,9	1,10	1,11	1,12	1,13	1,14	1,2
2	0,2	1,2	0,14	1,14	2,4	2,5	2,6	2,7	2,8	2,9	2,10	2,11	2,12	2,13	2,4	1,3
3	0,3	1,3	1,14	0,13	1,13	2,13	3,6	3,7	3,8	3,9	3,10	3,11	3,12	3,6	2,5	1,4
4	0,4	1,4	2,4	1,13	0,12	1,12	2,12	3,12	4,8	4,9	4,10	4,11	4,8	3,7	2,6	1,5
5	0,5	1,5	2,5	2,13	1,12	0,11	1,11	2,11	3,11	4,11	5,10	5,10	4,9	3,8	2,7	1,6
6	0,6	1,6	2,6	3,6	2,12	1,11	0,10	1,10	2,10	3,10	4,10	5,10	4,10	3,9	2,8	1,7
7	0,7	1,7	2,7	3,7	3,12	2,11	1,10	0,9	1,9	2,9	3,9	4,9	4,11	3,10	2,9	1,8
8	0,8	1,8	2,8	3,8	4,8	3,11	2,10	1,9	0,8	1,8	2,8	3,8	4,8	3,11	2,10	1,9
9	0,9	1,9	2,9	3,9	4,9	4,11	3,10	2,9	1,8	0,7	1,7	2,7	3,7	3,12	2,11	1,10
10	0,10	1,10	2,10	3,10	4,10	5,10	4,10	3,9	2,8	1,7	0,6	1,6	2,6	3,6	2,12	1,11
11	0,11	1,11	2,11	3,11	4,11	5,10	5,10	4,9	3,8	2,7	1,6	0,5	1,5	2,5	2,13	1,12
12	0,12	1,12	2,12	3,12	4,8	4,9	4,10	4,11	4,8	3,7	2,6	1,5	0,4	1,4	2,4	1,13
13	0,13	1,13	2,13	3,6	3,7	3,8	3,9	3,10	3,11	3,12	3,6	2,5	1,4	0,3	1,3	1,14
14	0,14	1,14	2,4	2,5	2,6	2,7	2,8	2,9	2,10	2,11	2,12	2,13	2,4	1,3	0,2	1,2
15	0,15	1,2	1,3	1,4	1,5	1,6	1,7	1,8	1,9	1,10	1,11	1,12	1,13	1,14	1,2	0,1

TABLE 4b. f odd

i \ j	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15
0	0,0	0,1	0,2	0,3	0,4	0,5	0,6	0,7	0,8	0,9	0,10	0,11	0,12	0,13	0,14	0,15
1	1,0	1,1	1,2	1,3	1,4	1,5	1,6	1,7	0,9	0,7	1,7	1,11	1,12	1,13	1,14	1,15
2	2,0	2,1	2,2	2,3	2,4	2,5	2,6	1,11	0,10	1,7	0,6	1,6	2,6	2,13	2,14	2,15
3	3,0	3,1	3,2	3,3	3,4	3,5	2,13	1,12	0,11	1,11	1,6	0,5	1,5	2,5	3,5	3,15
4	4,0	4,1	4,2	4,3	4,0	3,15	2,14	1,13	0,12	1,12	2,6	1,5	0,4	1,4	2,4	3,4
5	3,3	4,3	5,2	5,2	4,1	3,0	2,15	1,14	0,13	1,13	2,13	2,5	1,4	0,3	1,3	2,3
6	2,2	3,2	4,2	5,2	4,2	3,1	2,0	1,15	0,14	1,14	2,14	3,5	2,4	1,3	0,2	1,2
7	1,1	2,1	3,1	4,1	4,3	3,2	2,1	1,0	0,15	1,15	2,15	3,15	3,4	2,3	1,2	0,1
8	0,0	1,0	2,0	3,0	4,0	3,3	2,2	1,1	0,0	1,0	2,0	3,0	4,0	3,3	2,2	1,1
9	1,0	0,15	1,15	2,15	3,15	3,4	2,3	1,2	0,1	1,1	2,1	3,1	4,1	4,3	3,2	2,1
10	2,0	1,15	0,14	1,14	2,14	3,5	2,4	1,3	0,2	1,2	2,2	3,2	4,2	5,2	4,2	3,1
11	3,0	2,15	1,14	0,13	1,13	2,13	2,5	1,4	0,3	1,3	2,3	3,3	4,3	5,2	5,2	4,1
12	4,0	3,15	2,14	1,13	0,12	1,12	2,6	1,5	0,4	1,4	2,4	3,4	4,0	4,1	4,2	4,3
13	3,3	3,4	3,5	2,13	1,12	0,11	1,11	1,6	0,5	1,5	2,5	3,5	3,15	3,0	3,1	3,2
14	2,2	2,3	2,4	2,5	2,6	1,11	0,10	1,7	0,6	1,6	2,6	2,13	2,14	2,15	2,0	2,1
15	1,1	1,2	1,3	1,4	1,5	1,6	1,7	0,9	0,7	1,7	1,11	1,12	1,13	1,14	1,15	1,0

Department of Mathematics
University of California, San Diego
La Jolla, California 92093

Department of Applied Physics and Information Sciences
University of California, San Diego
La Jolla, California 92093

1. P. BARRUCAND & H. COHN, "Note on primes of the form $x^2 + 32y^2$, class number, and residuacity," *J. Reine Angew. Math.*, v. 238, 1969, pp. 67–70.

2. L. BAUMERT & H. FREDERICKSEN, "The cyclotomic numbers of order eighteen with applications to difference sets," *Math. Comp.*, v. 21, 1967, pp. 204–219.

3. B. C. BERNDT & R. J. EVANS, "Sums of Gauss, Jacobi, and Jacobsthal," *J. Number Theory*. (To appear.)

4. B. C. BERNDT & R. J. EVANS, "Sums of Gauss, Eisenstein, Jacobi, Jacobsthal, and Brewer," *Illinois J. Math.* (To appear.)

5. L. E. DICKSON, "Cyclotomy, higher congruences, and Waring's problem," *Amer. J. Math.*, v. 57, 1935, pp. 391–424.

6. R. J. EVANS, "Bioctic Gauss sums and sixteenth power residue difference sets," *Acta Arith.* (To appear.)

7. R. J. EVANS, "Resolution of sign ambiguities of Jacobi and Jacobsthal sums," *Pacific J. Math.* (To appear.)

8. R. GIUDICI, J. MUSKAT & S. ROBINSON, "On the evaluation of Brewer's character sums," *Trans. Amer. Math. Soc.*, v. 171, 1972, pp. 317–347.

9. H. HASSE, "Der 2^n-te Potenzcharakter von 2 im Körper der 2^n-ten Einheitswurzeln," *Rend. Circ. Mat. Palermo Ser.* II, v. 7, 1958, pp. 185–243.

10. E. LEHMER, "On the cyclotomic numbers of order sixteen," *Canad. J. Math.*, v. 6, 1954, pp. 449–454.

11. E. LEHMER, "On the number of solutions of $u^k + D \equiv w^2 \pmod{p}$," *Pacific J. Math.*, v. 5, 1955, pp. 103–118.

12. E. LEHMER, "On Jacobi functions," *Pacific J. Math.*, v. 10, 1960, pp. 887–893.

13. P. A. LEONARD & K. S. WILLIAMS, "The cyclotomic numbers of order eleven," *Acta Arith.*, v. 26, 1975, pp. 367–383.

14. P. A. LEONARD & K. S. WILLIAMS, "The cyclotomic numbers of order seven," *Proc. Amer. Math. Soc.*, v. 51, 1975, pp. 295–300.

15. J. B. MUSKAT, "The cyclotomic numbers of order fourteen," *Acta Arith.*, v. 11, 1966, pp. 263–279.

16. J. B. MUSKAT & A. L. WHITEMAN, "The cyclotomic numbers of order twenty," *Acta Arith.*, v. 17, 1970, pp. 185–216.

17. A. L. WHITEMAN, "The cyclotomic numbers of order sixteen," *Trans. Amer. Math. Soc.*, v. 86, 1957, pp. 401–413.

18. A. L. WHITEMAN, "The cyclotomic numbers of order ten," *Proc. Sympos. Appl. Math.*, vol. 10, Amer. Math. Soc., Providence, R. I., 1960, pp. 95–111.

19. A. L. WHITEMAN, "The cyclotomic numbers of order twelve," *Acta Arith.*, v. 6, 1960, pp. 53–76.