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Chapter 7 of Ramanujan's second notebook

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The principal topics in Chapter 7 of Ramanujan's second notebook concern sums of powers, an extended definition of Bernoulli numbers, the Riemann zeta-function $\zeta(s)$ and allied functions, Ramanujan's theory of divergent series, and the gamma function. This chapter thus represents a continuation of the subject matter of Chapters 5 and 6, which have been thoroughly examined elsewhere [6], [3]. Chapter 7 offers a considerable amount of numerical calculation. The extent of Ramanujan's calculations is amazing, since he evidently performed them without the aid of a mechanical or electrical device.

This paper provides proofs of the 110 theorems, formulas, and examples arranged in 27 sections of Chapter 7. The content of this chapter is, for the most part, correct, although there are several minor errors. We shall usually state and prove results for complex values of a variable. This is in contrast to Ramanujan who evidently intended his variables to be real. However, to give rigorous proofs, we have frequently needed to use analytic continuation, and so we state theorems in more generality than originally intended. It should be mentioned that we adhere to Ramanujan's designations of corollaries, examples, etc; these designations are frequently not optimal.

Several of the results in Chapter 7 are not new. For example, Ramanujan rediscovered the functional equation of $\zeta(s)$, found in Entry 4 in a somewhat disguised form. As was the case with Euler, Ramanujan had no real proof. It is fascinating how he arrived at this result, with reasoning based on his tenuous theory of the "constant" of a series.

Before commencing our examination of the individual results, we offer a few remarks about notation and state the Euler-Maclaurin summation formula. We shall adhere to the even suffixed notation for the Bernoulli number B_n and Euler numbers E_n, $0 \leqslant n < \infty$, as found in [1, p. 804], for example. These conventions are in contrast to those employed by Ramanujan. Thus, we define the Bernoulli numbers B_n, $0 \leqslant n < \infty$, by

$$x/(e^x - 1) = \sum_{k=0}^{\infty} B_k x^k/k! \quad |x| < 2\pi. \tag{1}$$

We shall very often use the Euler-Maclaurin summation formula. If f has $2n+1$ continuous derivatives on $[\alpha, \beta]$, where α and β are integers, then [13, p. 328], [31, Chapter 13]

$$\sum_{k=\alpha}^{\beta} f(k) = \int_{\alpha}^{\beta} f(t)\mathrm{d}t + \frac{1}{2}\{f(\alpha)+f(\beta)\} + \sum_{k=1}^{n} \frac{B_{2k}}{(2k)!}\{f^{(2k-1)}(\beta) - f^{(2k-1)}(\alpha)\} + R_n, \tag{2}$$

where

$$R_n = \int_\alpha^\beta P_{2n+1}(t) f^{(2n+1)}(t)\mathrm{d}t, \qquad n \geqslant 0,$$

where P_k denotes the kth periodic Bernoulli function.

We denote complex variables by s with $\sigma = \mathrm{Re}(s)$, by r with $u = \mathrm{Re}(r)$, and by z.

ENTRY 1

Let

$$\varphi_r(x) = \sum_{k=1}^{x} k^r, \tag{3}$$

where r is any complex number. Then if $r \neq -1$, as x tends to ∞,

$$\varphi_r(x) \sim \zeta(-r) + \frac{x^{r+1}}{r+1} + \frac{x^r}{2} + \sum_{k=1}^{\infty} \frac{B_{2k}\Gamma(r+1)x^{r-2k+1}}{(2k)!\Gamma(r-2k+2)}. \tag{4}$$

Proof. Applying the Euler-Maclaurin summation formula (2) with $f(t) = t^r$, $\alpha = 1$, and $\beta = x$, we find that

$$\varphi_r(x) = c + \frac{x^{r+1}}{r+1} + \frac{x^r}{2} + \sum_{k=1}^{n} \frac{B_{2k}\Gamma(r+1)x^{r-2k+1}}{(2k)!\Gamma(r-2k+2)} - \frac{\Gamma(r+1)}{\Gamma(r-2n)}\int_x^\infty P_{2n+1}(t)t^{r-2n-1}\mathrm{d}t,$$

where

$$c = -\frac{1}{r+1} + 1/2 - \sum_{k=1}^{n} \frac{B_{2k}\Gamma(r+1)}{(2k)!\Gamma(r-2k+2)} + \frac{\Gamma(r+1)}{\Gamma(r-2n)} \times \int_1^\infty P_{2n+1}(t)t^{r-2n-1}\mathrm{d}t \tag{5}$$

and n is a positive integer with $2n > u$. Note that

$$\int_x^\infty P_{2n+1}(t)t^{r-2n-1}\mathrm{d}t = O\left(\int_x^\infty t^{u-2n-1}\mathrm{d}t\right) = O(x^{u-2n}),$$

as $x \to \infty$. Thus, it remains to show that

$$c = \zeta(-r). \tag{6}$$

From the Euler-Maclaurin formula (2), we have, for $\sigma > 1$,

$$\zeta(s) = \frac{1}{s-1} + \frac{1}{2} + \sum_{k=1}^{n} \frac{B_{2k}\Gamma(s+2k-1)}{(2k)!\Gamma(s)} - \frac{\Gamma(s+2n+1)}{\Gamma(s)}\int_1^\infty P_{2n+1}(t)t^{-s-2n-1}\mathrm{d}t. \tag{7}$$

By analytic continuation, (7) holds for $\sigma + 2n > 0$. Putting $s = -r$ in (7) and using (5), we obtain (6).

The analogue of (4) for $r = -1$ is due to Euler [13, pp. 324, 325] and is stated by Ramanujan in Chapter 8, Entry 2 [44, vol. 2, p. 91], [7].

If r is a nonnegative integer, the series in (4) is finite and we may replace the asymptotic sign by an equality sign. Moreover, (4) reduces to the familiar formula

$$\varphi_r(x) = \frac{B_{r+1}(x+1) - B_{r+1}(1)}{r+1}, \tag{8}$$

where $B_n(x)$ denotes the nth Bernoulli polynomial.

In Chapter 5, section 25 [44, p. 57], [6], Ramanujan defines a Bernoulli number B_r^* of arbitrary index by

$$\zeta(r) = \frac{(2\pi)^r}{2\Gamma(r+1)} B_r^*. \tag{9}$$

In particular, if $r = 2n$ is an even integer, $B_{2n}^* = (-1)^{n-1} B_{2n}$ and (9) reduces to Euler's famous formula for $\zeta(2n)$. Using the functional equation for $\zeta(s)$ (see Entry 4), we find that

$$\zeta(-r) = \frac{B_{r+1}^* \cos\{\pi(r+1)/2\}}{r+1}. \tag{10}$$

In Ramanujan's version of (4), $\zeta(-r)$ is replaced by the right side of (10).

After Entry 1, Ramanujan makes some remarks about the "constant" of a series. This concept was introduced by Ramanujan in Chapter 6 and is discussed in detail in [3]. The "constant" in Entry 1 is merely the constant term $\zeta(-r)$ in the asymptotic expansion (4).

Now define, for $\sigma > 0$,

$$\eta(s) = \sum_{k=1}^{\infty} (-1)^{k+1} k^{-s}. \tag{11}$$

Note that

$$\eta(s) = (1 - 2^{1-s})\zeta(s), \tag{12}$$

which, by analytic continuation, is valid for all complex values of s.

ENTRY 2

For each complex number r,

$$\eta(-r) = \frac{(2^{r+1} - 1) B_{r+1}^* \sin(\pi r/2)}{r+1}.$$

Proof. Set $s = -r$ in (12) and use (10).

We now wish to extend the definition of $\phi_r(x)$ to encompass complex values of x. First, for $u < 0$, redefine

$$\varphi_r(x) = \sum_{k=1}^{\infty} \{k^r - (k+x)^r\}. \tag{13}$$

Observe that, if $u < -1$,

$$\varphi_r(x) = \zeta(-r) - \psi(-r, x+1), \tag{14}$$

where $\psi(-r, x+1) = \sum_{k=1}^{\infty} (k+x)^r$. Note that $\psi(s, x)$ is very closely related to the Hurwitz zeta-function $\zeta(s, x)$, except that the latter function is usually defined only for $0 < x \leqslant 1$. The methods for analytically continuing $\zeta(s, x)$ (see, e.g., [2], [48, p. 37], [50, p. 268]) normally can be easily adapted to establish the analytic continuation of $\psi(s, x)$ as well. Thus, by analytic continuation, we shall now define $\varphi_r(x)$, for all complex values of x and r, by (14). Moreover, if $u < 0$ and x is a positive integer, we find from (13) that

$$\varphi_r(x) = \sum_{k=1}^{x} k^r. \tag{15}$$

By analytic continuation, (15) is valid for all complex values of r. Thus, the new definition (14) agrees with our former definition (3) if x is a positive integer. If r is a nonnegative integer and $-1 < x \leqslant 0$, then by (14) and the well-known fact $\psi(-r, x+1) = -B_{r+1}(x+1)/(r+1)$, we find that

$$\varphi_r(x) = \frac{B_{r+1}(x+1) - B_{r+1}(1)}{r+1}.$$

By analytic continuation, this holds for all x, and so we see that (14) is in agreement with (8) if r is a nonnegative integer and x is complex.

Corollary

If r is complex and $r \neq -1$, then

$$\varphi_r(-1/2) = \frac{(2-2^{-r})B^*_{r+1}\cos\{\pi(r+1)/2\}}{r+1}.$$

Proof. By (13), if $u < 0$, we easily find that

$$\varphi_r(-1/2) = -2^{-r}\eta(-r), \tag{16}$$

where $\eta(s)$ is defined by (11). By analytic continuation, (16) is valid for all complex r, $r \neq -1$. Using Entry 2 in (16), we complete the proof.

If r is a positive integer, then, by (8), the Corollary is equivalent to the well-known fact $B_{r+1}(1/2) = -(1-2^{-r})B_{r+1}$ [1, p. 805].

Entry 3

Let $\varphi_r(x)$ be defined by (14) and let a and b be complex numbers with $b \neq 0$. Then

$$\sum_{k=1}^{x} (a+kb)^r = b^r\{\varphi_r(x+a/b) - \varphi_r(a/b)\}.$$

Proof. For $r < 0$, the desired formula follows easily from (13). The result for all r follows by analytic continuation.

Entry 4

For any complex number r.

$$\frac{\sin(\pi r/2)B^*_{1-r}}{1-r} = \zeta(r) = \frac{(2\pi)^r}{2\Gamma(r+1)}B^*_r. \tag{17}$$

Proof. We present Ramanujan's interesting argument, which is not rigorous.

Rewriting (8) in Ramanujan's notation, we have

$$\varphi_r(x) = \frac{B_{r+1}(x+1) - \sin(\pi r/2)B^*_{r+1}}{r+1},$$

where r is a natural number. We now suppose that this formula is valid for *all* r. The "constant" in this representation for $\varphi_{-r}(x)$ is

$$\frac{\sin(\pi r/2)B^*_{1-r}}{1-r}.$$

On the other hand, from (6) and (9), the "constant" is also equal to

$$\zeta(r) = \frac{(2\pi)^r B^*_r}{2\Gamma(r+1)}.$$

These constants must be equal, and hence (17) follows.

The equalities in (17) imply that

$$\zeta(r) = 2(2\pi)^{r-1}\Gamma(1-r)\zeta(1-r)\sin(\pi r/2). \tag{18}$$

Mirabile dictu, Ramanujan has derived the functional equation of $\zeta(r)$ [48, p. 25] in a most unorthodox manner!

Corollary 1

We have $B^*_{-2} = 2\zeta(3)$, $B^*_{-4} = -4\zeta(5)$, $B^*_{-6} = 6\zeta(7)$, and $B^*_{-8} = -8\zeta(9)$.

Proof. The proposed equalities are special instances of the first equality in (17).

Corollary 2

$$\Gamma(1/2) = \sqrt{\pi}.$$

Corollary 3

For every complex number z, $\Gamma(z)\Gamma(1-z) = \pi/\sin(\pi z)$.

Corollary 2 is the special case $z = 1/2$ of the well-known Corollary 3 for which we give Ramanujan's proof.

Proof. Letting $r = -z$ and $r = z+1$ in the extremal sides of (17), we find, respectively, that

$$-\frac{\sin(\pi z/2)B^*_{z+1}}{z+1} = \frac{(2\pi)^{-z}B^*_{-z}}{2\Gamma(1-z)}$$

and

$$-\frac{\cos(\pi z/2)B^*_{-z}}{z} = \frac{(2\pi)^{z+1}B^*_{z+1}}{2\Gamma(z+2)}.$$

Multiplying these two equalities together, using the equality $\Gamma(z+2) = (z+1)z\Gamma(z)$, and simplifying, we obtain the desired result.

Corollary 4

We have

$$\pi \sum_{k=0}^{\infty} \frac{(-1)^k}{(2k)^{1/2} + (2k+2)^{1/2}} = \sum_{k=0}^{\infty} 1/(2k+1)^{3/2}.$$

Ramanujan gives essentially the following faulty proof of Corollary 4. From (11) with $s = -1/2$, (12), and (18), it follows that

$$\pi \sum_{k=0}^{\infty} (-1)^k/((2k)^{1/2} + (2k+2)^{1/2}) = \frac{\pi}{\sqrt{2}} \sum_{k=0}^{\infty} (-1)^k \{(k+1)^{1/2} - \sqrt{k}\}$$

$$= \pi\sqrt{2} \sum_{k=0}^{\infty} (-1)^{k+1} \sqrt{k} = \pi\sqrt{2}(1 - 2^{3/2})\zeta(-1/2)$$

$$= (1 - 2^{-3/2})\zeta(3/2) = \sum_{k=0}^{\infty} \frac{1}{(2k+1)^{3/2}}.$$

Corollary 4, in fact, is the special case $p = 1/2$ of the identity

$$\pi^{p+1} \sum_{k=1}^{\infty} (-1)^k \{(k+1)^p - k^p\} = 4 \sin(\pi p/2)\Gamma(p+1) \sum_{k=0}^{\infty} \frac{1}{(2k+1)^{p+1}},$$

proved in [47, p. 154], where $0 < p < 1$.

Ramanujan [41], [43, pp. 47–49] has established some other results which are akin to Corollary 4. Kesava Menon [34] has given simpler proofs of Ramanujan's results and has proved additional results of this type as well.

Corollary 5

Let $\eta(s)$ be defined by (11). Then

$$(2\pi)^{2/3}\eta(1/3) = (1 + 2^{1/3})\Gamma(2/3)\eta(2/3).$$

Proof. Using (12), we rewrite the functional equation (18) for $\zeta(r)$ in terms of $\eta(r)$ to get

$$(1 - 2^r)\eta(r) = 2(2\pi)^{r-1}(1 - 2^{1-r})\sin(\pi r/2)\Gamma(1-r)\eta(1-r).$$

Putting $r = 1/3$ and simplifying, we achieve the desired result.

Corollary 6

As $x \to \infty$,

$$\sum_{k=1}^{x} 1/\sqrt{k} \sim 2\sqrt{x} + \zeta(1/2) + \frac{1}{2\sqrt{x}} - \sum_{k=1}^{\infty} \frac{B_{2k}}{(2k)!} \frac{1\cdot 3 \ldots (4k-3)}{2^{2k-1}} x^{-2k+1/2}. \tag{19}$$

This asymptotic formula is a special case of (4) but is different from that claimed by Ramanujan [44, vol. 2, p. 79] who asserts that, as $x \to \infty$,

$$\sum_{k=1}^{x} 1/\sqrt{k} \sim (2 + 4x)^{1/2} + \zeta(1/2). \tag{20}$$

Formulas (19) and (20) are incompatible since

$$(2+4x)^{1/2} = 2\sqrt{x} + \frac{1}{2\sqrt{x}} - \frac{1}{16x^{3/2}} + \ldots;$$

the leading two terms above agree with (19), but the third term does not coincide with the corresponding term $-x^{-3/2}/24$ in (19). Ramanujan gives no indication as to how he arrived at the approximation $(2+4x)^{1/2}$.

COROLLARY 7

As $x \to \infty$,

$$\sum_{k=1}^{x} \sqrt{k} \sim \frac{2}{3}x^{3/2} + \frac{1}{2}x^{1/2} - \frac{1}{4\pi}\zeta(3/2) + \frac{1}{24}x^{-1/2} + \sum_{k=2}^{\infty} \frac{B_{2k}}{(2k)!} \frac{1\cdot 3\ldots(4k-5)}{2^{2k-1}} x^{-2k+3/2}. \tag{21}$$

Using the functional equation (18), we observe that Corollary 7 is a special instance of Entry 1, but (21) is not the result claimed by Ramanujan. Instead, he has proposed that

$$\sum_{k=1}^{x} \sqrt{k} \sim \frac{2}{3}[(x+1/4)(x+1/2)(x+3/4)]^{1/2} - \frac{1}{4\pi}\zeta(3/2), \tag{22}$$

as $x \to \infty$. Since the infinite series in (21) diverges while

$$\frac{2}{3}[(x+1/4)(x+1/2)(x+3/4)]^{1/2} = \frac{2}{3}x^{3/2} + \frac{1}{2}x^{1/2} + \frac{1}{24}x^{-1/2} + \ldots$$

converges in a neighbourhood of $x = \infty$, (21) and (22) are certainly not compatible. However, note that the right side of (22) does provide a good approximation for the left side.

A similar type of approximation for $\varphi_{1/2}(x)$ has been obtained by Gates [22].

COROLLARY 8

As $x \to \infty$,

$$\sum_{k=1}^{x} k^{3/2} \sim \frac{2}{5}x^{5/2} + \frac{1}{2}x^{3/2} + \frac{1}{8}x^{1/2} - \frac{3}{16\pi^2}\zeta(5/2) - 3\sum_{k=2}^{\infty} \frac{B_{2k}}{(2k)!} \frac{1\cdot 3\ldots(4k-7)}{2^{2k-1}} x^{-2k+5/2}. \tag{23}$$

Corollary 8 follows from Entry 1 and the functional equation (18). In contrast to (23), Ramanujan claims that

$$\sum_{k=1}^{x} k^{3/2} \sim \frac{2}{5}(x(x+1/4)(x+1/2)(x+3/4)(x+1) + \frac{5}{768}(x+1/2))^{1/2} - \frac{3}{16\pi^2}\zeta(5/2), \tag{24}$$

as $x \to \infty$. By the same reasoning used in conjunction with Corollary 7, (23) and (24) are incompatible. However,

$$\frac{2}{5}(x(x+1/4)(x+1/2)(x+3/4)(x+1)$$
$$+\frac{5}{768}(x+1/2))^{1/2} = \frac{2}{5}x^{5/2}+\frac{1}{2}x^{3/2}+\frac{1}{8}x^{1/2}+\cdots.$$

ENTRY 5

Let a and b be complex numbers with $b \neq 0$ and a/b not a negative integer. Then if $u < -1$,

$$\sum_{k=1}^{\infty}(-1)^{k+1}(a+kb)^r = (2b)^r\left\{\varphi_r\left(\frac{a}{2b}\right)-\varphi_r\left(\frac{a-b}{2b}\right)\right\}. \tag{25}$$

Proof. We have

$$\sum_{k=1}^{\infty}(-1)^{k+1}(a+kb)^r = \sum_{k=1}^{\infty}(a+(2k-1)b)^r - \sum_{k=1}^{\infty}(a+2kb)^r,$$

and the desired equality follows immediately from the definition (13).

There is a misprint in the notebooks [44, vol. 2, p. 79]; Ramanujan has written b^r instead of $(2b)^r$ on the right side of (25).

ENTRY 6(i)

Let x be a positive integer and assume that $n > 0$. Then

$$(x^2+x)^n = 2\sum_{k=0}^{\infty}\binom{n}{2k+1}\varphi_{2n-2k-1}(x).$$

Proof. The proof is indicated by Ramanujan. Expanding by the binomial theorem, we have, for $|z| \geqslant 1$ and $n > 0$,

$$(z^2+z)^n-(z^2-z)^n = z^{2n}\{(1+1/z)^n-(1-1/z)^n\}$$
$$= z^{2n}\left\{\sum_{k=0}^{\infty}\binom{n}{k}z^{-k}-\sum_{k=0}^{\infty}\binom{n}{k}(-z)^{-k}\right\}$$
$$= 2z^{2n}\sum_{k=0}^{\infty}\binom{n}{2k+1}z^{-2k-1}. \tag{26}$$

Now set $z = j$ in (26) and sum both sides on j, $1 \leqslant j \leqslant x$, to get

$$(x^2+x)^n = 2\sum_{j=1}^{x}j^{2n}\sum_{k=0}^{\infty}\binom{n}{2k+1}j^{-2k-1}.$$

The required formula follows upon inverting the order of summation and employing equation (3).

ENTRY 6(ii)

Under the same hypotheses as Entry 6(i), we have

$$(x+1/2)(x^2+x)^n = \sum_{k=0}^{\infty}\left\{2\binom{n}{2k+1}+\binom{n}{2k}\right\}\varphi_{2n-2k}(x).$$

Proof. Proceeding in the same fashion as in the previous proof, we find that, for $|z| \geqslant 1$ and $n > 0$,

$$(z+1/2)(z^2+z)^n-(z-1/2)(z^2-z)^n$$
$$= z^{2n}\sum_{k=0}^{\infty}\left\{2\binom{n}{2k+1}+\binom{n}{2k}\right\}z^{-2k}. \tag{27}$$

Letting $z=j$ in (27) and summing both sides on j, $1 \leqslant j \leqslant x$, we arrive at the formula that we sought with no difficulty.

Note that if n is a positive integer, then Entries 6(i) and 6(ii) are valid for all x because they yield polynomial identities.

COROLLARY 1

Let $y = x^2+x$ and $a = x+1/2$. Then

$$\varphi_1(x) = \tfrac{1}{2}y, \qquad \varphi_2(x) = \tfrac{1}{3}ay, \qquad \varphi_3(x) = \tfrac{1}{4}y^2,$$
$$\varphi_4(x) = \tfrac{1}{5}ay(y-\tfrac{1}{3}), \qquad \varphi_5(x) = \tfrac{1}{6}y^2(y-\tfrac{1}{2}),$$
$$\varphi_6(x) = \tfrac{1}{7}ay(y^2-y+\tfrac{1}{3}), \qquad \varphi_7(x) = \tfrac{1}{8}y^2(y^2-\tfrac{4}{3}y+\tfrac{2}{3}),$$
$$\varphi_8(x) = \tfrac{1}{9}ay(y^3-2y^2+\tfrac{9}{5}y-\tfrac{3}{5}),$$
$$\varphi_9(x) = \tfrac{1}{10}y^2(y-1)(y^2-\tfrac{3}{2}y+\tfrac{3}{2}),$$
$$\varphi_{10}(x) = \tfrac{1}{11}ay(y-1)(y^3-\tfrac{7}{3}y^2+\tfrac{10}{3}y-\tfrac{5}{3}),$$

and

$$\varphi_{11}(x) = \tfrac{1}{12}y^2(y^4-4y^3+\tfrac{17}{2}y^2-10y+5).$$

Proof. The proposed odd indexed formulas for $\varphi_r(x)$ follow from Entry 6(i) by successively letting $n = 1, 2, \ldots, 6$. The proposed even indexed formulas arise from Entry 6(ii) by successively setting $n = 1, 2, \ldots, 5$.

Although the formulas in Corollary 1 have long been known and are instances of (8), Ramanujan's method for determining them by means of Entries 6(i) and 6(ii) is particularly brief and elegant. For other formulas and methods for finding $\varphi_r(x)$ when r and x are positive integers, see a survey paper by Snow [45] which contains several references.

COROLLARY 2

For each positive integer n, we have

(i) $$\sum_{k=1}^{n}\left(\frac{2k-1+\sqrt{5}}{9}\right)^9 = \varphi_9\left(\frac{2n-1+\sqrt{5}}{2}\right)$$

and

(ii) $$\sum_{k=1}^{n}\left(\frac{2k-1+\sqrt{5}}{2}\right)^{10} = \varphi_{10}\left(\frac{2n-1+\sqrt{5}}{2}\right).$$

If p and n are positive integers with n even, then

(iii) $$\sum_{k=1}^{p}(2k-1)^n = 2^n\varphi_n(p-1/2).$$

Proof. Applying Entry 3 with $r = 9, x = n, a = (\sqrt{5}-1)/2$, and $b = 1$, we find that

$$\sum_{k=1}^{n}\left(\frac{2k-1+\sqrt{5}}{2}\right)^{9} = \varphi_9\left(\frac{2n-1+\sqrt{5}}{2}\right) - \varphi_9\left(\frac{\sqrt{5}-1}{2}\right).$$

However, by Corollary 1, it is easily seen that $(\sqrt{5}-1)/2$ is a root of $\varphi_9(x)$. Hence, part (i) follows.

Part (ii) follows in the same fashion as part (i), except now we use the fact that $\varphi_{10}((\sqrt{5}-1)/2) = 0$.

To prove (iii), apply Entry 3 with $r = n$, $x = p$, $a = -1$, and $b = 2$ to get

$$\sum_{k=1}^{p} (2k-1)^n = 2^n\{\varphi_n(p-1/2) - \varphi_n(-1/2)\}.$$

By the Corollary to Entry 2, $\varphi_n(-\frac{1}{2}) = 0$, and the proof is complete.

ENTRY 7

If r is a positive integer, then

$$\varphi_r(x-1) + (-1)^r\varphi_r(-x) = 0.$$

Proof. By (8),

$$\varphi_r(x-1) + (-1)^r\varphi_r(-x) = \frac{B_{r+1}(x) + (-1)^r B_{r+1}(1-x)}{r+1}.$$

By a very familiar property of Bernoulli polynomials [1, p. 804], the right side above is equal to 0.

COROLLARY

If r is a positive integer exceeding 1, then $\varphi_r(x)$ is divisible by $x^2(x+1)^2$ or $x(x+\frac{1}{2})(x+1)$ according as r is odd or even.

Proof. This result follows easily from Entries 6(i), (ii) by induction on r.

ENTRY 8

If r is a positive integer, then

$$\begin{aligned}\varphi_r(x) &= \frac{1}{r+1}\sum_{k=0}^{r}\binom{r+1}{k}B_k x^{r+1-k} + x^r \\ &= \frac{x^{r+1}}{r+1} + \frac{x^r}{2} - 2\sum_{k=1}^{[r/2]}\binom{r+1}{2k}k\zeta(1-2k)x^{r+1-2k}. \qquad (28)\end{aligned}$$

Proof. Using the well-known formula [1, p. 804]

$$B_r(x) = \sum_{k=0}^{r}\binom{r}{k}B_k x^{r-k}, r \geqslant 0,$$

in (8), we find that

$$\varphi_r(x) = \varphi_r(x-1) + x^r = \frac{1}{r+1}\sum_{k=0}^{r+1}\binom{r+1}{k}B_k x^{r+1-k} - \frac{B_{r+1}}{r+1} + x^r,$$

from which the first equality of (28) readily follows. The latter equality of (28) follows from (17).

ENTRY 9. is simply a restatement of (13).

ENTRY 10.

For each complex number r and each positive integer n,

$$\varphi_r(x) - n^r \sum_{k=0}^{n-1} \varphi_r\left(\frac{x-k}{n}\right) = (1 - n^{r+1})\zeta(-r)$$
$$= (n^{r+1} - 1)\frac{\sin(\pi r/2)B_{r+1}^*}{r+1}. \tag{29}$$

Proof. For $u < -1$, (13) yields

$$\varphi_r(x) - n^r \sum_{k=0}^{n-1} \varphi_r\left(\frac{x-k}{n}\right)$$
$$= \zeta(-r) - \sum_{j=1}^{\infty} (j+x)^r - n^{r+1}\zeta(-r) + \sum_{k=0}^{n-1}\sum_{j=1}^{\infty} (nj-k+x)^r$$
$$= (1 - n^{r+1})\zeta(-r). \tag{30}$$

By analytic continuation, the extremal sides of (30) are equal for all r. The second equality in (29) follows from (17).

COROLLARY

Under the hypotheses of Entry 10,

$$\sum_{k=1}^{n-1} \varphi_r(-k/n) = (n - n^{-r})\zeta(-r).$$

Proof. Put $x = 0$ in Entry 10 and use the fact that $\varphi_r(0) = 0$ for each r.

ENTRY 11

If r is a positive integer, then

$$\varphi_{-r}(x-1) + (-1)^r \varphi_{-r}(-x)$$
$$= \{1 + (-1)^r\}\zeta(r) + \frac{(-1)^r}{(r-1)!}\frac{d^{r-1}}{dx^{r-1}}(\pi\cot(\pi x)), \tag{31}$$

where if $r = 1$, the first expression on the right side of (31) is understood to be equal to 0.

Proof. By (13),

$$\varphi_{-r}(x-1) + (-1)^r \varphi_{-r}(-x)$$
$$= \{1 + (-1)^r\}\zeta(r) - \sum_{k=0}^{\infty}\left\{\frac{1}{(k+x)^r} + \frac{1}{(x-k-1)^r}\right\}.$$

Since

$$\pi \cot(\pi x) = \sum_{k=0}^{\infty} \left\{ \frac{1}{k+x} + \frac{1}{x-k-1} \right\},$$

equality (31) now easily follows.

In the notebooks [44, vol. 2, p. 81], Ramanujan gives (31) with r replaced by $-r$ and states that the result is obtained by differentiating the equality $\varphi_{-1}(x-1) - \varphi_{-1}(-x) = -\pi \cot(\pi x)$ r times. The correct number of differentiations is however $-r-l$. Ramanujan then indicates that (31) holds for negative as well as positive values of r and that Entry 7 can thus be deduced. If we interpret

$$\frac{(-1)^r}{(r-1)!} \frac{\mathrm{d}^{r-1}}{\mathrm{d}x^{r-1}} \cot(\pi x)$$

as being identically 0 for $r < 0$, then, indeed, we obtain Entry 7 (but this does *not prove* Entry 7).

Ramanujan next indicates a method for calculating the derivatives of $\cot(\pi x)$. We are not certain what Ramanujan's method is, but it seems to be a more complicated version of the simple method which we describe below. This method has been judiciously applied and generalized by Carlitz and Scoville [14], [15]. Set $y = \cot(\pi x)$. Then $\tan^{-1}(1/y) = \pi x$. Upon differentiating both sides of the latter equality with respect to x, we find that

$$\frac{\mathrm{d}y}{\mathrm{d}x} = -\pi(y^2+1). \tag{32}$$

Further derivatives of $\cot(\pi x)$ can be found by successively differentiating (32). In this manner, the following table of derivatives of $\cot(\pi x)$ may rapidly be calculated. All formulas are correctly given by Ramanujan, except that he has written 2385 for 2835 in the last denominator of the last entry.

k	$\frac{(-1)^k}{\pi^k k!} \frac{\mathrm{d}^k y}{\mathrm{d}x^k}$
0	y
1	y^2+1
2	y^3+y
3	$y^4+\frac{4}{3}y^2+\frac{1}{3}$
4	$y^5+\frac{5}{3}y^3+\frac{2}{3}y$
5	$y^6+2y^4+\frac{17}{15}y^2+\frac{2}{15}$
6	$y^7+\frac{7}{3}y^5+\frac{77}{45}y^3+\frac{17}{45}y$
7	$y^8+\frac{8}{3}y^6+\frac{12}{5}y^4+\frac{248}{315}y^2+\frac{17}{315}$
8	$y^9+3y^7+\frac{16}{5}y^5+\frac{88}{63}y^3+\frac{62}{315}y$
9	$y^{10}+\frac{10}{3}y^8+\frac{37}{9}y^6+\frac{424}{189}y^4+\frac{1382}{2835}y^2+\frac{62}{2835}$

COROLLARY

If r is any complex number, then

(i) $$\varphi_r(x) - 2^r\left\{\varphi_r\left(\frac{x}{2}\right) + \varphi_r\left(\frac{x-1}{2}\right)\right\} = (1 - 2^{r+1})\zeta(-r),$$

(ii) $$\varphi_r(-1/2) = (2 - 2^{-r})\zeta(-r),$$

(iii) $$\varphi_r(-1/3) + \varphi_r(-2/3) = (3 - 3^{-r})\zeta(-r),$$

(iv) $$\varphi_r(-1/4) + \varphi_r(-3/4) = (2 + 2^{-r} - 4^{-r})\zeta(-r),$$

and

(v) $$\varphi_r(-1/6) + \varphi_r(-5/6) = (1 + 2^{-r} + 3^{-r} - 6^{-r})\zeta(-r).$$

Proof. Part (i) is the case $n = 2$ of Entry 10. Parts (ii)–(v) follow from the Corollary to Entry 10 by successively setting $n = 2$, 3, 4 and 6, respectively.

EXAMPLES

If r is a positive, odd integer, then

(i) $$\varphi_r(-1/3) = (3 - 3^{-r})\zeta(-r)/2,$$

(ii) $$\varphi_r(-1/4) = (1 + 2^{-r-1} - 2^{-2r-1})\zeta(-r),$$

(iii) $$\varphi_r(-1/6) = (1 + 2^{-r} + 3^{-r} - 6^{-r})\zeta(-r)/2,$$

(iv) $$\varphi_r(-1/5) + \varphi_r(-2/5) = (5 - 5^{-r})\zeta(-r)/2,$$

(v) $$\varphi_r(-1/8) + \varphi_r(-3/8) = (2 + 2^{-2r-1} - 2^{-3r-1})\zeta(-r),$$

(vi) $$\varphi_r(-1/10) + \varphi_r(-3/10) = (3 + 2^{-r} + 5^{-r} - 10^{-r})\zeta(-r)/2,$$

and

(vii) $$\varphi_r(-1/12) + \varphi_r(-5/12) = (4 - 2^{-r} + 4^{-r} + 6^{-r} - 12^{-r})\zeta(-r)/2.$$

Proof. All of these formulas are easily established with the use of the Corollary of Entry 10 and Entry 7. For illustration, we shall give the proof of part (vii). By the aforementioned results,

$$(12 - 12^{-r})\zeta(-r) = \sum_{k=1}^{11} \varphi_r(-k/12) = 2\sum_{k=1}^{5} \varphi_r(-k/12) + \varphi_r(-1/2).$$

Using Examples (i), (ii) and (iii) and Corollary (ii), we find that

$$\begin{aligned}\varphi_r(-1/12) + \varphi_r(-5/12) &= (12 - 12^{-r})\zeta(-r)/2 \\ &- (3 - 3^{-r})\zeta(-r)/2 - (1 + 2^{-r-1} - 2^{-2r-1})\zeta(-r) \\ &- (1 + 2^{-r} + 3^{-r} - 6^{-r})\zeta(-r)/2 - (2 - 2^{-r})\zeta(-r)/2,\end{aligned}$$

which, upon simplification, yields (vii).

Ramanujan has incorrectly given the right sides of (vi) and (vii) [44, vol. 2, p. 83]. The examples above are more commonly expressed in terms of values of Bernoulli polynomials. For example, see [1, pp. 805, 806].

ENTRY 12

For every complex number r,

$$2^r\{\varphi_r(-1/6)-\varphi_r(-5/6)\} = (2^r+1)\{\varphi_r(-1/3)-\varphi_r(-2/3)\}. \tag{33}$$

Proof. Putting $n=2$ and $x=-1/3$ and $x=-2/3$ in Entry 10, we find that, respectively,

$$\varphi_r(-1/3)-2^r\{\varphi_r(-1/6)+\varphi_r(-2/3)\} = (1-2^{r+1})\zeta(-r) \tag{34}$$

and

$$\varphi_r(-2/3)-2^r\{\varphi_r(-1/3)+\varphi_r(-5/6)\} = (1-2^{r+1})\zeta(-r). \tag{35}$$

Subtracting (35) from (34) and rearranging terms, we deduce (33).

The proof above is given by Ramanujan in the notebooks, but he has inadvertently multiplied the right sides of (34) and (35) by -1.

EXAMPLE 1

$$\sum_{k=0}^{\infty}\frac{1}{(2k+1)^3} = \frac{7}{8}\zeta(3).$$

EXAMPLE 2

$$\sum_{k=0}^{\infty}\frac{1}{(3k+1)^3} = \frac{2\pi^3}{81\sqrt{3}}+\frac{13}{27}\zeta(3).$$

EXAMPLE 3

$$\sum_{k=0}^{\infty}\frac{1}{(4k+1)^3} = \frac{\pi^3}{64}+\frac{7}{16}\zeta(3).$$

EXAMPLE 4

$$\sum_{k=0}^{\infty}\frac{1}{(6k+1)^3} = \frac{\pi^3}{36\sqrt{3}}+\frac{91}{216}\zeta(3).$$

All of these examples follow from well-known general formulas. Example 1 is trivial. Examples 2–4 follow from general formulas for $\varphi_{-2n-1}(-2/3)$, $\varphi_{-2n-1}(-3/4)$, and $\varphi_{-2n-1}(-5/6)$ that can be found in Hansen's tables [28, formulas (6.3.10), (6.3.18), and (6.3.23), pp. 118, 119].

ENTRY 13

For each nonnegative integer k, define

$$c_k = \lim_{m\to\infty}\left(\sum_{j=1}^{m}\frac{\log^k j}{j}-\frac{\log^{k+1} m}{k+1}\right). \tag{36}$$

Then for all s,

$$\zeta(s) = \frac{1}{s-1}+\sum_{k=0}^{\infty}\frac{(-1)^k c_k}{k!}(s-1)^k. \tag{37}$$

In particular, if $A_k = (-1)^k c_k/k!$, $0 \leqslant k < \infty$, then

$$A_0 = \gamma = 0{\cdot}5772156649,$$
$$A_1 = 0{\cdot}0728158455, \quad A_2 = -0{\cdot}00485, \quad \text{and} \quad A_3 = -0{\cdot}00034, \tag{38}$$

where γ denotes Euler's constant.

Ramanujan did not explicitly define c_k by (36). Instead, he says that c_k is the constant of $\Sigma_{j=1}^{\infty} (\log^k j/j)$, but this is equivalent to (36). The values of A_k, $0 \leqslant k \leqslant 3$, are correct to the given number of decimal places.

The Laurent series (37) has been independently discovered several times in the literature. Apparently, Stieltjes [46] first established (37) in 1885. Furthermore, Stieltjes and Hermite [46, letters 73, 74, 75, 77] have thoroughly discussed this result in an exchange of letters. Not surprisingly, the constants A_k are now called Stieltjes constants. In 1887, Jensen [33] rediscovered (37). Hardy [30], [32, pp. 475–476] and Ramanujan [42], [43, p. 134] himself each stated (37) without proof. Briggs and Chowla [12] rediscovered (37) again in 1955. Later proofs have been given by Verma [49] and Ferguson [19] in 1963 and Lammel [38] in 1966. Kluyver [36] has established an infinite series representation for c_k.

Wilton [51] and Berndt [2] have evaluated the Laurent coefficients of the Hurwitz zeta-function. Further generalizations to other Dirichlet series have been found by Briggs and Buschman [11] and Knopfmacher [37].

Numerical calculations of the constants c_k were first carried out by Jensen [33] who calculated the first 9 coefficients to 9 decimal places. In 1895, Gram [26] published a table of the first 16 coefficients to 16 decimal places. The most extensive calculations todate have been by Liang and Todd [39] who calculated the first 20 coefficients to 15 decimal places.

Briggs [10] and Mitrović [40] have proved theorems on the signs of the coefficients c_k. Uniform bounds for $|c_k|$ have been established by Briggs [10]; the best estimates to date are due to Berndt [2].

Example 1

For $|n|$ sufficiently small, we have

$$\zeta(1+n)+\zeta(1-n) = \frac{2\gamma}{1+0{\cdot}00839\,n^2-0{\cdot}0001\,n^4+\ldots}.$$

Proof. From Entry 13, for $|n|$ sufficiently small,

$$\zeta(1+n)+\zeta(1-n) = 2\gamma+2A_2n^2+2A_4n^4+\ldots$$
$$= \frac{2\gamma}{1-\dfrac{A_2}{\gamma}n^2+\left(\dfrac{A_2^2}{\gamma^2}-\dfrac{A_4}{\gamma}\right)n^4+\ldots}.$$

Using the values of A_0, A_2 and A_4 given by Liang and Todd [39] and employing a calculator, we complete the proof.

In Example 1 Ramanujan, in fact, has written $+0{\cdot}0001\,n^4$ instead of $-0{\cdot}0001\,n^4$. Several of the following examples also need corrections.

EXAMPLE 2

$$\zeta(11/10) = 10{\cdot}58444842.$$

EXAMPLE 3

$$\zeta(3/2) = 2{\cdot}6123752.$$

EXAMPLE 4

$$\zeta(5/2) = 1{\cdot}341490.$$

EXAMPLE 5

$$B^*_{3/2} = 0{\cdot}4409932; \; B^*_{1/2} = -1{\cdot}032627.$$

EXAMPLE 6

$$B^*_{1/3} = -0{\cdot}9420745; \; B^*_{-1/3} = -1{\cdot}3841347.$$

EXAMPLE 7

$$B^*_{-1/2} = -1{\cdot}847228.$$

According to Gram's [27] table of values for $\zeta(s)$ which has been reproduced in Dwight's tables [18], the last recorded digit for $\zeta(11/10)$ should be 6 rather than 2. These same tables indicate that the last recorded digit for $\zeta(3/2)$ is 3 and not 2 and that the last two digits of $\zeta(5/2)$ are 87 instead of 90. In an earlier table of Glaisher [23], the values of $\zeta(11/10)$ and $\zeta(3/2)$ are found to six decimal places.

The five particular values of B^*_r given by Ramanujan can be found by employing (17) in conjunction with tabulated values of the Riemann zeta-function. Using the value of $\zeta(3/2)$, we find that the last digit of $B^*_{3/2}$ should be 3 rather than 2. The given values of $B^*_{1/2}$ and $B^*_{-1/2}$ are correct. To calculate $B^*_{1/3}$ and $B^*_{-1/3}$ we need the values of $\zeta(2/3)$ and $\zeta(4/3)$ which are not found in the aforementioned tables but which have been calculated by Hansen and Patrick [29]. Accordingly, the last digit of $B^*_{1/3}$ should be 3 rather than 5. Ramanujan's value of $B^*_{-1/3}$, in contrast to his other calculations, is somewhat off from the correct value $-1{\cdot}3860016$.

For a list of all tables of the Riemann zeta-function before 1962, consult the Index of Fletcher *et al* [20]. The most extensive computations of $\zeta(s)$ appear to have been done by McLellan in 1968; see Wrench's review [52] for a description of these tables.

ENTRY 14

Let $n > 0$. Then as $n \to 0$,

$$\sum_{k=2}^{\infty} \frac{1}{k(k^n - 1)} \sim \frac{a_0 - \log n}{n} + a_1 + \sum_{k=1}^{\infty} a_{k+1} n^{2k-1}, \tag{39}$$

where

$$a_0 = \lim_{m \to \infty} \left(\sum_{k=2}^{m} \frac{1}{k \log k} - \log \log m \right) = 0{\cdot}7946786, \tag{40}$$

$$a_1 = \tfrac{1}{2}(1 - \gamma) = 0{\cdot}2113922,$$

and

$$a_{k+1} = \frac{-B_{2k}A_{2k-1}}{2k}, \; k \geqslant 1,$$

where B_j denotes the jth Bernoulli number and A_j is defined in Entry 13, $0 \leqslant j < \infty$. In particular, $a_2 = -0{\cdot}0060680$ and $a_3 = -0{\cdot}000000475$.

The numerical value for $a_0 + \log\log 2$ is found in an article of Boas [9, p. 156]. Boas records the first six digits of $a_0 + \log\log 2$ in [8, p. 244]. The numerical values for a_1, a_2, and a_3 may be determined from (38), or, more accurately, from the table of Liang and Todd [39]. While Ramanujan correctly gives a_1 and a_2, his value $-0{\cdot}0000028$ for a_3 is incorrect.

Proof. Let $t \geqslant 1$, $x \geqslant 0$, and suppose that $0 < n < A$, where A is fixed and positive but otherwise arbitrary. Define

$$f(t) = \frac{1}{t(t^n-1)}, \; h(t) = f(t) - \frac{1}{nt\log t}, \text{ and } g(x) = \frac{1}{e^x-1} - \frac{1}{x}.$$

Then $h(t) = g(n\log t)/t$ and

$$h'(t) = t^{-2}\{ng'(n\log t) - g(n\log t)\}. \tag{41}$$

Fix an integer $N \geqslant 1$. Applying Taylor's theorem to g and g', we see from (41) that

$$\begin{aligned} t^2h'(t) = &\sum_{j=0}^{N} \frac{(n\log t)^j}{j!}\{ng^{(j+1)}(0) - g^{(j)}(0)\} \\ &+ \frac{(n\log t)^{N+1}}{(N+1)!}\{ng^{(N+2)}(\theta_1) - g^{(N+1)}(\theta_2)\}, \end{aligned} \tag{42}$$

where $0 \leqslant \theta_1, \theta_2 \leqslant n\log t$. By the definition (1) for the Bernoulli numbers,

$$g^{(j)}(0)/j! = B_{j+1}/(j+1)!, \quad j \geqslant 0. \tag{43}$$

Thus, as $x \to 0$, $g^{(N)}(x) \to B_{N+1}/(N+1)$. Using the fact that

$$g(x) = -\frac{1}{x} + \sum_{k=0}^{\infty} \exp(-(k+1)x), \quad x > 0,$$

we find that $g^{(N)}(x) \to 0$ as $x \to \infty$. Hence, $g^{(N)}(x)$ is bounded for each fixed N, and so the last expression in (42) is $0(\{n\log t\}^{N+1})$, where the implied constant is independent of n and t. Using (43), we deduce from (42) that

$$\begin{aligned} t^2h'(t) &= \sum_{j=0}^{N} \frac{g^{(j)}(0)}{j!}\{nj(n\log t)^{j-1} - (n\log t)^j\} + 0(\{n\log t\}^{N+1}) \\ &= t^2 \sum_{j=0}^{N} \frac{B_{j+1}n^j}{(j+1)!}\frac{d}{dt}\left(\frac{\log^j t}{t}\right) + 0(\{n\log t\}^{N+1}). \end{aligned} \tag{44}$$

Next, apply the Euler-Maclaurin formula (2) with 0, $h(t)$, 1, and m playing the roles of n, $f(t)$, α, and β, respectively. Since $h(1) = -1/2$, we find that

$$\sum_{k=2}^{m} h(k) = \int_1^m h(t)\,dt + 1/4 + \frac{h(m)}{2} + \int_1^m P_1(t)h'(t)\,dt. \tag{45}$$

From (44) and (45), it follows that

$$\sum_{k=2}^{m} h(k) = \int_1^m h(t)\,\mathrm{d}t + 1/4 + \frac{h(m)}{2} + \sum_{j=0}^{N} \frac{B_{j+1} n^j}{(j+1)!} \int_1^m P_1(t) \frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\log^j t}{t}\right)\mathrm{d}t + O(n^{N+1}), \tag{46}$$

where the implied constant is independent of n and m. We now evaluate the integrals on the right side of (46). First,

$$\int_1^m h(t)\,\mathrm{d}t = \frac{1}{n}\log\left(\frac{t^n - 1}{t^n \log t}\right)\Bigg|_1^m = \frac{1}{n}\log\left(\frac{m^n-1}{m^n}\right) - \frac{\log\log m}{n} - \frac{\log n}{n}. \tag{47}$$

By the Euler-Maclaurin formula (2),

$$\int_1^m P_1(t)\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\log^j t}{t}\right)\mathrm{d}t$$

$$= \begin{cases} \displaystyle\sum_{k=1}^{m} \frac{\log^j k}{k} - \int_1^m \frac{\log^j t}{t}\,\mathrm{d}t - \frac{\log^j m}{2m} = \sum_{k=1}^{m}\frac{\log^j k}{k} - \frac{\log^{j+1} m}{j+1} - \frac{\log^j m}{2m}, & j > 0, \\ \displaystyle\sum_{k=1}^{m} \frac{1}{k} - \int_1^m \frac{\mathrm{d}t}{t} - \frac{1}{2}\left(1 + \frac{1}{m}\right) = \sum_{k=1}^{m}\frac{1}{k} - \log m - \frac{1}{2}\left(1+\frac{1}{m}\right), & j = 0. \end{cases} \tag{48}$$

Using the integral evaluations (47) and (48) in (46) and then letting $m \to \infty$, we find that

$$\sum_{k=2}^{\infty} f(k) = \frac{a_0}{n} - \frac{\log n}{n} + 1/4 + B_1(\gamma - 1/2) + \sum_{j=1}^{N} \frac{B_{j+1} n^j c_j}{(j+1)!} + O(n^{N+1}),$$

where c_j is defined in (36). The asymptotic formula (39) now readily follows.

Corollary 1 is a restatement of (40).

Corollary 2

For $s > 0$,

$$\sum_{k=2}^{\infty} \frac{1}{k^{s+1}\log k} = -\log s + C + (1-\gamma)s - \sum_{k=2}^{\infty} \frac{A_{k-1} s^k}{k}, \tag{49}$$

where

$$C = \sum_{k=2}^{\infty} \frac{1}{k^2 \log k} - 1 + \gamma + \sum_{k=2}^{\infty} \frac{A_{k-1}}{k}, \tag{50}$$

and where A_k, $1 \leqslant k < \infty$, is defined in Entry 13. Furthermore, $C = 0{\cdot}2174630$, $1-\gamma = 0{\cdot}4227843$, $-\frac{1}{2}A_1 = -0{\cdot}0364079$, $-\frac{1}{3}A_2 = 0{\cdot}001615$, $-\frac{1}{4}A_3 = 0{\cdot}000086$, and $-\frac{1}{5}A_4 = -0{\cdot}00002$.

Proof. Replacing s by $x+1$ in (37) and integrating over $[1, s]$, we find that, for $s > 0$,

$$-\sum_{k=2}^{\infty}\frac{1}{k^{s+1}\log k}+\sum_{k=2}^{\infty}\frac{1}{k^2\log k}+s-1$$
$$=\int_1^s \zeta(x+1)\,\mathrm{d}x=\log s+\gamma(s-1)+\sum_{k=2}^{\infty}\frac{A_{k-1}(s^k-1)}{k}.$$

Hence, (49) and (50) follow immediately.

The numerical coefficients of s^k, $1\leqslant k\leqslant 5$, are now found by employing the table of Liang and Todd [39]. The value

$$\sum_{k=2}^{\infty}\frac{1}{k^2\log k}=0{\cdot}605521788883$$

was calculated by J R Hill on his PDP11/34 computer. Using this computation along with the table from [39] and the bounds $|A_k|\leqslant 4/(k\pi^k)$, $1\leqslant k<\infty$, found in a paper of Berndt [2], we derive the proposed value of C.

Ramanujan's version of Corollary 2 contains some minor discrepancies; his coefficients of s^3 and s^4 are 0·001617 and 0·000085, respectively.

Entry 15

Let $u>-1$ and $0<x<1$. Then

$$\frac{\varphi_r(x-1)-\varphi_r(-x)}{4\Gamma(r+1)}=-\cos(\pi r/2)\sum_{k=1}^{\infty}\frac{\sin(2\pi kx)}{(2\pi k)^{r+1}}.$$

Proof. Recall Hurwitz's formula [48, p. 37]

$$\zeta(s,a)=2\Gamma(1-s)\left\{\sin(\pi s/2)\sum_{k=1}^{\infty}\frac{\cos(2\pi ka)}{(2\pi k)^{1-s}}+\cos(\pi s/2)\sum_{k=1}^{\infty}\frac{\sin(2\pi ka)}{(2\pi k)^{1-s}}\right\},\tag{51}$$

where $\sigma<1$ and $0<a\leqslant 1$. By (14) and (51), the desired formula readily follows.

Entry 16

Let $u>-1$ and $0<x<1$. Then

$$\frac{\varphi_r(x-1)+\varphi_r(-x)-2\zeta(-r)}{4\Gamma(r+1)}=\sin(\pi r/2)\sum_{k=1}^{\infty}\frac{\cos(2\pi kx)}{(2\pi k)^{r+1}}.$$

Proof. The proof is completely analogous to the previous proof.

Corollary 16(i)

Let p and q be integers with $0<p<q$. Then if r is any complex number,

$$\frac{(2\pi q)^r}{4\Gamma(r)}\left\{\varphi_{r-1}\left(\frac{p}{q}-1\right)-\varphi_{r-1}\left(-\frac{p}{q}\right)\right\}$$
$$=-\sin(\pi r/2)\sum_{j=1}^{q-1}\sin(2\pi jp/q)\left\{\zeta(r)-\varphi_{-r}\left(\frac{j}{q}-1\right)\right\}.\tag{52}$$

Proof. Using Entry 15 and putting $k=mq+j$, $1\leqslant j\leqslant q$, $0\leqslant m<\infty$, we find that, for $r>1$,

$$\frac{(2\pi q)^r}{4\Gamma(r)}\left\{\varphi_{r-1}\left(\frac{p}{q}-1\right)-\varphi_{r-1}\left(-\frac{p}{q}\right)\right\} = -\sin(\pi r/2)q^r\sum_{k=1}^{\infty}\frac{\sin(2\pi kp/q)}{k^r}$$

$$= -\sin(\pi r/2)\sum_{j=1}^{q}\sin(2\pi jp/q)\sum_{m=0}^{\infty}\frac{1}{(m+j/q)^r}.$$

The result now follows from (14) for $r > 1$ and by analytic continuation for all r.

COROLLARY 16(ii)

Let p and q be integers with $0 < p < q$. For any complex number r,

$$\frac{(2\pi q)^r}{4\Gamma(r)}\left\{\varphi_{r-1}\left(\frac{p}{q}-1\right)+\varphi_{r-1}\left(-\frac{p}{q}\right)-2(1-q^{-r})\zeta(1-r)\right\}$$

$$= -\cos(\pi r/2)\sum_{j=1}^{q-1}\cos(2\pi jp/q)\left\{\zeta(r)-\varphi_{-r}\left(\frac{j}{q}-1\right)\right\}. \quad (53)$$

Proof. The proof is similar to the previous proof, but, in addition, uses the functional equation (18).

For the next few results we shall need Ramanujan's extended concept of the Euler numbers (see [6, section 25]). Define

$$E_r^* = \frac{2\Gamma(r)}{(\pi/2)^r}L(r) \quad (54)$$

where r is any complex number, and where

$$L(s) = \sum_{k=0}^{\infty}(-1)^k(2k+1)^{-s}, \quad \sigma > 0. \quad (55)$$

It is well-known [16, Chapter 9] that $L(s)$ can be analytically continued to an entire function. Observe that $E_{2n+1}^* = (-1)^n E_{2n}$, where n is a non-negative integer and E_{2j} denotes the $2j$th Euler number, which is defined by

$$\sec x = \sum_{j=0}^{\infty}\frac{(-1)^j E_{2j}x^{2j}}{(2j)!}, \quad |x| < \pi/2.$$

ENTRY 17

For each complex number r,

$$\varphi_r(-1/4)-\varphi_r(-3/4) = \frac{2\cos(\pi r/2)E_{r+1}^*}{4^{r+1}}.$$

Proof. Put $x = 1/4$ in Entry 15 to get, for $u > -1$,

$$\frac{\varphi_r(-3/4)-\varphi_r(-1/4)}{4\Gamma(r+1)} = -\cos(\pi r/2)\sum_{k=1}^{\infty}\frac{\sin(\pi k/2)}{(2\pi k)^{r+1}}$$

$$= -\frac{\cos(\pi r/2)}{(2\pi)^{r+1}}L(r+1)$$

$$= -\frac{\cos(\pi r/2)E_{r+1}^*}{4^{r+1}2\Gamma(r+1)},$$

by (55) and (54). The desired result now follows for $u > -1$. By analytic continuation, the proposed formula is valid for all r.

COROLLARY

For $u < 0$,

$$\sum_{k=0}^{\infty} (-1)^k (2k+1)^r = \frac{1}{2}\cos(\pi r/2) E^*_{r+1}.$$

Proof. By Entry 17 and (13), with $u < 0$,

$$\frac{2\cos(\pi r/2) E^*_{r+1}}{4^{r+1}} = \varphi_r(-1/4) - \varphi_r(-3/4)$$

$$= \sum_{k=1}^{\infty} \{(k-3/4)^r - (k-1/4)^r\}$$

$$= 4^{-r} \sum_{k=0}^{\infty} (-1)^k (2k+1)^r,$$

and the result follows.

ENTRY 18

For each complex number r,

$$\cos(\pi r/2) E^*_{1-r} = 2L(r) = (\pi/2)^r E^*_r / \Gamma(r). \tag{56}$$

The equalities in (56) yield the functional equation of $L(r)$ [16, p. 69],

$$L(r) = \cos(\pi r/2)(\pi/2)^{r-1}\Gamma(1-r)L(1-r).$$

Proof. We present here Ramanujan's argument.
By (54), the "constant" for the series $L(r)$ is

$$L(r) = \frac{(\pi/2)^r}{2\Gamma(r)} E^*_r.$$

But by the last corollary, the "constant" for $L(r)$ is also equal to

$$\tfrac{1}{2}\cos(\pi r/2) E^*_{1-r}.$$

Since these two constants must be equal, (56) follows at once.

Ramanujan's derivation of the next corollary was evidently very similar to his argument for Corollary 4 in section 4.

COROLLARY

We have

$$\pi\left(1/2 + \sum_{k=1}^{\infty} \frac{(-1)^k}{(2k-1)^{1/2} + (2k+1)^{1/2}}\right) = L(3/2).$$

Proof. Since the proof is very similar to that of Corollary 4 in section 4, we shall present only a brief sketch. If we replace $f(x)$ by $f(x+\pi/2)$ in [47, pp. 153–154], we find that

$$\pi^{p+1}+\pi^{p+1}\sum_{k=1}^{\infty}(-1)^k\{(2k+1)^p-(2k-1)^p\}$$
$$=2^{p+2}\cos(\pi p/2)\Gamma(p+1)L(p+1),$$

where $0<p<1$, after a completely analogous argument. Putting $p=1/2$, we complete the proof of the Corollary.

Entry 19(i)

Assume the hypotheses of Corollary 16(i) with the additional assumption that q is odd. Then

$$\frac{(2\pi q)^r}{4\Gamma(r)}\left\{\varphi_{r-1}\left(\frac{p}{q}-1\right)-\varphi_{r-1}\left(-\frac{p}{q}\right)\right\}$$
$$=\sin(\pi r/2)\sum_{j=1}^{(q-1)/2}\sin(2\pi jp/q)\left\{\varphi_{-r}\left(\frac{j}{q}-1\right)-\varphi_{-r}\left(-\frac{j}{q}\right)\right\}.$$

Proof. On the right side of (16.1) replace j by $q-j$ in that part of the sum with $(q+1)/2\leqslant j\leqslant q-1$.

Entry 19(ii)

Suppose that all hypotheses of Corollary 16(ii) hold. Assume also that q is odd. Then

$$\frac{(2\pi q)^r}{4\Gamma(r)}\left\{\varphi_{r-1}\left(\frac{p}{q}-1\right)+\varphi_{r-1}\left(-\frac{p}{q}\right)-2\zeta(1-r)\right\}$$
$$=\cos(\pi r/2)\sum_{j=1}^{(q-1)/2}\cos(2\pi jp/q)\left\{\varphi_{-r}\left(\frac{j}{q}-1\right)+\varphi_{-r}\left(-\frac{j}{q}\right)\right\}.$$

Proof. Using (53), proceed in the same fashion as in the previous proof. In addition, the functional equation (18) must be employed.

Ramanujan's version of Entry 19(ii) is incorrect [44, vol. 2, pp. 86, 87].

Corollary 1

Let $u>0$ and suppose that $0\leqslant x<1$. Then

$$\frac{2^{r-1}\pi^{r+1}}{\Gamma(r+1)}\varphi_r(-x)=\sum_{k=1}^{\infty}\frac{\sin(\pi kx)\cos(\pi kx+\pi r/2)}{k^{r+1}}.$$

Proof. By (14), (18) and (51), we find that for $u>0$ and $0\leqslant x<1$,

$$\frac{2^{r-1}\pi^{r+1}}{\Gamma(r+1)}\varphi_r(-x)=\frac{2^{r-1}\pi^{r+1}}{\Gamma(r+1)}\{\zeta(-r)-\zeta(-r,1-x)\}$$
$$=-\frac{1}{2}\sin(\pi r/2)\zeta(r+1)+\frac{1}{2}\sin(\pi r/2)$$
$$\times\sum_{k=1}^{\infty}\frac{\cos(2\pi kx)}{k^{r+1}}+\frac{1}{2}\cos(\pi r/2)\sum_{k=1}^{\infty}\frac{\sin(2\pi kx)}{k^{r+1}}$$
$$=\frac{1}{2}\sum_{k=1}^{\infty}\frac{1}{k^{r+1}}\{-\sin(\pi r/2)+\sin(2\pi kx+\pi r/2)\}$$
$$=\sum_{k=1}^{\infty}\frac{\sin(\pi kx)\cos(\pi kx+\pi r/2)}{k^{r+1}},$$

upon using the identity $-\sin A + \sin(A+2B) = 2\sin B\cos(A+B)$. This completes the proof.

COROLLARY 2

If $0 < x < 1$, then

$$\sum_{k=0}^{\infty}\left(\frac{1}{(k+x)^{1/2}} - \frac{1}{(k+1-x)^{1/2}}\right) = 2\sum_{k=1}^{\infty}\frac{\sin(2\pi kx)}{\sqrt{k}}.$$

Proof. Set $r = -1/2$ in Entry 15 and use (13).

ENTRY 20

If r is any complex number, then

$$\frac{(6\pi)^r}{2\sqrt{3}\Gamma(r)}\{\varphi_{r-1}(-1/3) - \varphi_{r-1}(-2/3)\}$$
$$= \sin(\pi r/2)\{\varphi_{-r}(-1/3) - \varphi_{-r}(-2/3)\}.$$

Proof. Put $p = 1$ and $q = 3$ in Entry 19(i), and the result follows.

Section 21 appears to have no relation to the other material in Chapter 7. In Entry 21, Ramanujan writes

$$\varphi_\infty\left(\frac{nx}{1+x}\right) = (1+x)^{-n}\sum_{k=0}^{\infty}\binom{n}{k}\varphi(k)x^k, \tag{57}$$

where

$$\varphi_r(x) = \varphi_{r-1}(x) - 1 + \exp\left(\frac{nP_{r-1}\varphi_{r-1}^r(x)}{r!}\right),$$

with $\varphi_1(x) = \varphi(x)$, and where

$$P_r = \sum_{k=1}^{\infty}(-1)^{k+1}k^r x^k.$$

We have been unable to discern the meaning of this result, since the recursively defined functions $\varphi_r(x)$ have not been connected with (57) in any way. We shall regard (57) as the definition of φ_∞. Setting $u = nx/(1+x)$ and $p = u/n$, we find that (57) becomes

$$\varphi_\infty(u) = \sum_{k=0}^{\infty}\binom{n}{k}\varphi(k)p^k(1-p)^{n-k}. \tag{58}$$

In the following corollary, Ramanujan gives a formula for $\varphi_\infty(u)$ in terms of the derivatives $\varphi^{(j)}(u)$, $0 \leqslant j < \infty$. Note that $\varphi_\infty(u)$ is the expected value of $\varphi(u)$ if u denotes a random variable with binomial distribution $b(n, k; p)$. Ramanujan alludes to Entry 10 of Chapter 3 [4], where he gives a formula for the expected value of $\varphi(u)$ in terms of $\varphi^{(j)}(u)$, $0 \leqslant j < \infty$, where u denotes a Poisson random variable. However, the latter result appears to be considerably deeper than the present corollary.

COROLLARY

Let u and n be fixed where $0 \leqslant u \leqslant n$ and n is an integer. Let $\varphi(z)$ be analytic in a disc centered at u and containing the segment $[0, n]$. If $\varphi_\infty(u)$ is defined by (58), then

$$\varphi_\infty(u) = \sum_{j=0}^{\infty} \frac{\varphi^{(j)}(u)}{j!} \sum_{k=0}^{\infty} \binom{n}{k} (k-u)^j p^k (1-p)^{n-k}. \tag{59}$$

Proof. Expanding $\varphi(z)$ in its Taylor series about u, we find that

$$\varphi_\infty(u) = \sum_{k=0}^{n} \binom{n}{k} \sum_{j=0}^{\infty} \frac{\varphi^{(j)}(u)}{j!} (k-u)^j p^k (1-p)^{n-k}.$$

The equality (59) now follows by inverting the order of summation.

Observe that the Corollary even holds when n is an arbitrary positive number, provided that $p < 1/2$ and φ is a polynomial. It would be interesting to find more general conditions under which (59) holds.

ENTRY 22

Let $A_1 = 0$. For each non-negative integer r, $r \neq 1$, set $A_r = \{1 + (-1)^r\}\zeta(r)$. If n is a natural number, then

$$\sum_{k=1}^{\infty} \frac{1}{k^n (k+1)^n} = \sum_{k=0}^{n} A_{n-k} \binom{-n}{k}.$$

Proofs of Entry 22 have been given by Glaisher [24], Kesava Menon [35], and Djoković [17]. Entry 22 is identical with Entry 35 in Chapter 9 [5]. The following example is the special case $n = 3$ of Entry 22.

EXAMPLE

$$\sum_{k=1}^{\infty} \frac{1}{k^3 (k+1)^3} = 10 - \pi^2.$$

Entry 23 offers the very well-known asymptotic expansion of $\log \Gamma(z+1)$ [50, p. 252].

ENTRY 23

Let $|\arg z| < \pi$. Then as $|z| \to \infty$.

$$\log \Gamma(z+1) \sim (z + 1/2)\log z - z + \frac{1}{2}\log(2\pi) + \sum_{k=1}^{\infty} \frac{B_{2k}}{2k(2k-1)z^{2k-1}}.$$

Ramanujan remarks that $\frac{1}{2}\log(2\pi) = 0{\cdot}918938533204673$, which is correct [1, p. 3].

We quote the following corollary exactly as it appears in the notebooks [44, vol. 2, p. 88]. This approximation for the gamma function is remindful of Corollaries 6–8 in section 4.

COROLLARY

When x is great $e^x \Gamma(x+1)/x^x = (2\pi x + \pi/3)^{1/2}$ nearly.

Proof. From the familiar asymptotic expansion for $\Gamma(x)$, as $x \to \infty$ [50, p. 253],

$$\frac{e^x \Gamma(x+1)}{x^x} \sim \sqrt{2\pi x}\left(1 + \frac{1}{12x} + \frac{1}{288x^2} + \dots\right). \tag{60}$$

But, on the other hand,

$$(2\pi x + \pi/3)^{1/2} = (2\pi x)^{1/2}\,(1 + 1/(6x))^{1/2}$$
$$= (2\pi x)^{1/2}\left(1 + \frac{1}{12x} - \frac{1}{288x^2} + \ldots\right). \qquad (61)$$

Thus, Ramanujan's approximation is reasonable, but observe that the coefficients of x^{-2} in (60) and (61) are of opposite sign.

Entry 24 and its corollary are restatements of Corollaries 3 and 2, respectively, of section 4.

Entry 25

For every complex number z and positive integer n,

$$\prod_{k=1}^{n} \Gamma\left(\frac{z+k}{n}\right) = (2\pi)^{(n-1)/2} n^{-z-1/2} \Gamma(z+1).$$

Entry 25 is a version of Gauss's famous multiplication theorem for the gamma function [50, p. 240]. Corollary 1 is the special case $z = 0$ of Entry 25.

Corollary 2

$$\Gamma(2/3) = [\Gamma(5/6)]^{1/2} 2^{1/3} (\pi/3)^{1/4}.$$

Proof. Put $n = 2$ and $z = -1/3$ in Entry 25 to get $\Gamma(1/3)\Gamma(5/6) = \sqrt{\pi}\, 2^{1/3}\, \Gamma(2/3)$. By Corollary 3 of section 4, $\Gamma(2/3)\Gamma(1/3) = 2\pi/\sqrt{3}$. Combining these two equalities, we achieve the desired result.

Corollary 3

For every complex number z,

$$\Gamma(z+1) = \Gamma\left(\frac{z+1}{2}\right)\Gamma\left(\frac{z}{2}+1\right)2^z\pi^{-1/2}.$$

Corollary 3 is Legendre's duplication formula and is the special case $n = 2$ of Gauss's multiplication formula, Entry 25.

Corollary 4

Let $|\arg z| < \pi$. Then as $|z| \to \infty$,

$$\log\Gamma(z+1/2) \sim z\log z - z + \frac{1}{2}\log(2\pi) + \sum_{k=1}^{\infty} \frac{B_{2k}(2^{1-2k}-1)}{2k(2k-1)z^{2k-1}}.$$

Proof. Replacing z by $2z$ in Corollary 3, we find that

$$\Gamma(z+1/2) = \frac{\sqrt{\pi}\,\Gamma(2z+1)}{2^{2z}\Gamma(z+1)}.$$

Take logarithms on both sides and apply Entry 23.

Ramanujan inadvertently multiplied the infinite series above by -1.

The Maclaurin series in Entry 26 is well-known [1, p. 256].

ENTRY 26

For $|z| < 1$,

$$\log \Gamma(z+1) = -\gamma z + \sum_{k=2}^{\infty} \frac{\zeta(k)(-z)^k}{k},$$

where γ denotes Euler's constant.

COROLLARY

For $|z| < 1$,

$$\begin{aligned}\log\{\tfrac{1}{2}\Gamma(z+3)\} = {} & 0{\cdot}9227843351z + 0{\cdot}1974670334z^2 \\ & - 0{\cdot}0256856344z^3 + 0{\cdot}0049558084z^4 \\ & - 0{\cdot}0011355510z^5 + 0{\cdot}0002863437z^6 \\ & - 0{\cdot}0000766825z^7 + 0{\cdot}0000213883z^8 \\ & - 0{\cdot}0000061409z^9 + 0{\cdot}0000018013z^{10} + \ldots . \end{aligned} \tag{62}$$

Proof. Using Entry 26, we find that, for $|z| < 1$,

$$\begin{aligned}\log\{\tfrac{1}{2}\Gamma(z+3)\} &= \log(\tfrac{1}{2}z+1) + \log(z+1) + \log\Gamma(z+1) \\ &= \sum_{k=1}^{\infty} \frac{(-1)^{k+1}}{k}\left(\frac{z}{2}\right)^k + \sum_{k=1}^{\infty} \frac{(-1)^{k+1} z^k}{k} \\ &\quad - \gamma z + \sum_{k=2}^{\infty} \frac{\zeta(k)(-z)^k}{k}.\end{aligned}$$

The given numerical values for the coefficients of z^k, $1 \leqslant k \leqslant 10$, now follow by direct calculation. Numerical values of $\zeta(k)$, $2 \leqslant k \leqslant 10$, may be found in [1, p. 811] or [18, p. 224].

In Ramanujan's formulation of (62), he replaces the tenth and all succeeding terms by the single expression $0{\cdot}0000054047\, z^{10}/(3+z)$.

Our calculations below were determined from the values of $\Gamma(2/3)$, $\Gamma(5/6)$, and $\Gamma(9/10)$ found in [21]. Ramanujan inexplicably gives the value 0·5341990853 for $\log \Gamma(2/3)$.

EXAMPLE 1

$$\log \Gamma(2/3) = 0{\cdot}3031502752.$$

EXAMPLE 2

$$\log \Gamma(5/6) = 0{\cdot}1211436313.$$

EXAMPLE 3

$$\log \Gamma(9/10) = 0{\cdot}0663762397.$$

ENTRY 27(i)

Suppose that n is a natural number and that $|z| > n$. Then

$$2\pi z \prod_{k=1}^{\infty}\left(1+\frac{z^2}{(n+k)^2}\right) = 2\left(\frac{n!}{z^n}\right)^2 \sinh(\pi z)\exp\left(\sum_{k=1}^{\infty}\frac{(-1)^k\varphi_{2k}(n)}{kz^{2k}}\right). \tag{63}$$

Proof. Using Euler's definition of the gamma function [50, p. 237]

$$\Gamma(z) = \lim_{k\to\infty}\frac{(k-1)!\,k^z}{z(z+1)\ldots(z+k-1)},$$

we find that

$$\frac{\Gamma^2(n+1)}{\Gamma(n+1+iz)\Gamma(n+1-iz)}$$
$$= \lim_{k\to\infty}\frac{\{(n+1)^2+z^2\}\{(n+2)^2+z^2\}\ldots\{(n+k)^2+z^2\}}{(n+1)(n+2)^2\ldots(n+k)^2}$$
$$= \prod_{k=1}^{\infty}\left(1+\frac{z^2}{(n+k)^2}\right). \tag{64}$$

On the other hand,

$$\frac{2\pi z\Gamma^2(n+1)}{\Gamma(n+1+iz)\Gamma(n+1-iz)}$$
$$= \frac{2\pi z\Gamma^2(n+1)}{z^{2n}(1+1^2/z^2)(1+2^2/z^2)\ldots(1+n^2/z^2)\Gamma(1+iz)\Gamma(1-iz)}. \tag{65}$$

Using the Maclaurin series for $\log(1+y)$ with $y = k^2/z^2$, $1 \leqslant k \leqslant n$, we find that

$$\prod_{k=1}^{n}(1+k^2/z^2)^{-1} = \exp\left(-\sum_{k=1}^{n}\log(1+k^2/z^2)\right)$$
$$= \exp\left(\sum_{j=1}^{\infty}\frac{(-1)^j\varphi_{2j}(n)}{jz^{2j}}\right), \tag{66}$$

provided that $|z| > n$. Also, by Corollary 3 of section 4,

$$\frac{2\pi z}{\Gamma(1+iz)\Gamma(1-iz)} = \frac{2\pi}{i\Gamma(iz)\Gamma(1-iz)} = 2\sinh(\pi z). \tag{67}$$

Now substitute (66) and (67) into (65). Comparing the resulting equality with (64), we readily deduce (63).

In Ramanujan's formulations of Entry 27 (i) and Entry 27 (ii) [44, vol. 2, pp. 89, 90], instead of $2\sinh(\pi x)$, there appears $e^{\pi x} - e^{-\pi x}\theta$, but $e^{-\pi x}\theta$ is struck out. In a footnote, which is also struck out, Ramanujan says that "$\theta = \cos 2\pi n$ exactly or very nearly according as $2n$ is an integer or not." A two line solution to Entry 27(i) is also crossed out.

Entry 27(ii)

Under the same hypotheses as Entry 27(i), we have

$$2\pi(z^2+n^2)^{n+1/2}\prod_{k=1}^{\infty}\left(1+\frac{z^2}{(n+k)^2}\right)$$
$$= 2(n!)^2\sinh(\pi z)\exp\left(2n - 2z\tan^{-1}(n/z) + \sum_{j=1}^{\infty}\frac{(-1)^jB_{2j}S_{2j}}{jz^{2j-1}}\right), \tag{68}$$

where

$$S_{2j} = \sum_{k=0}^{\infty} \frac{(-1)^k (2k+2j-1)!}{(2j-1)!(2k+1)!} \left(\frac{n}{z}\right)^{2k+1}, j \geq 1.$$

Proof. First,

$$(z^2+n^2)^{n+1/2}/z = z^{2n} \exp[(n+1/2)\log(1+n^2/z^2)]. \tag{69}$$

Multiplying both sides of (63) by $(z^2+n^2)^{n+1/2}/z$ and utilizing (69), we find that, for $|z| > n$,

$$2\pi(z^2+n^2)^{n+1/2} \prod_{k=1}^{\infty} \left(1+\frac{z^2}{(n+k)^2}\right)$$

$$= 2(n!)^2 \sinh(\pi z) \exp\left((n+1/2)\sum_{k=1}^{\infty} \frac{(-1)^{k+1} n^{2k}}{kz^{2k}} + \sum_{k=1}^{\infty} \frac{(-1)^k \varphi_{2k}(n)}{kz^{2k}}\right). \tag{70}$$

Comparing (68) with (70), we see that it remains to show that

$$(n+1/2)\sum_{k=1}^{\infty} \frac{(-1)^{k+1} n^{2k}}{kz^{2k}} + \sum_{k=1}^{\infty} \frac{(-1)^k \varphi_{2k}(n)}{kz^{2k}}$$

$$= 2n - 2z\tan^{-1}(n/z) + \sum_{j=1}^{\infty} \frac{(-1)^j B_{2j} S_{2j}}{jz^{2j-1}}. \tag{71}$$

By Entry 1 and the remarks prior to (8), since $\zeta(-2k) = 0$, $k \geq 1$,

$$\sum_{k=1}^{\infty} \frac{(-1)^k \varphi_{2k}(n)}{kz^{2k}} = \sum_{k=1}^{\infty} \frac{(-1)^k n^{2k+1}}{k(2k+1)z^{2k}} + \sum_{k=1}^{\infty} \frac{(-1)^k n^{2k}}{2kz^{2k}}$$

$$+ \sum_{k=1}^{\infty} \frac{(-1)^k}{kz^{2k}} \sum_{j=1}^{k} \frac{B_{2j}(2k)! n^{2k-2j+1}}{(2j)!(2k-2j+1)!}. \tag{72}$$

Now a short calculation shows that

$$\sum_{k=1}^{\infty} \frac{(-1)^k n^{2k+1}}{k(2k+1)z^{2k}} + \sum_{k=1}^{\infty} \frac{(-1)^k n^{2k}}{2kz^{2k}} + (n+1/2)\sum_{k=1}^{\infty} \frac{(-1)^{k+1} n^{2k}}{kz^{2k}}$$

$$= 2z\sum_{k=1}^{\infty} \frac{(-1)^{k+1} n^{2k+1}}{(2k+1)z^{2k+1}} = 2n - 2z\tan^{-1}(n/z).$$

Thus, we only need yet to examine the double sum in (72). Inverting the order of summation by absolute convergence, we find that this double series becomes

$$\sum_{j=1}^{\infty} \frac{B_{2j}}{(2j)!} \sum_{k=j}^{\infty} \frac{(-1)^k (2k)! n^{2k-2j+1}}{k(2k-2j+1)! z^{2k}}$$

$$= \sum_{j=1}^{\infty} \frac{(-1)^j B_{2j}}{z^{2j-1}} \sum_{\mu=0}^{\infty} \frac{(-1)^\mu (2\mu+2j)!}{(2j)!(\mu+j)(2\mu+1)!} \left(\frac{n}{z}\right)^{2\mu+1}$$

After a slight amount of simplification, the double series above is easily seen to be equal to the series on the right side of (71). This completes the proof of (71) and hence of (68).

ENTRY 27(iii).

Let n be a positive integer and suppose that $x > 0$. Write $r^2 = n^2 + x^2$ with $r > 0$ and put $\beta = \tan^{-1}(x/n)$. Then as $x \to \infty$,

$$\log\left\{2\pi(x^2+n^2)^{n-1/2}\prod_{k=0}^{\infty}\left(1+\frac{x^2}{(n+k)^2}\right)\right\}$$

$$\sim 2\log\Gamma(n)+2n+2x\beta-\sum_{k=1}^{\infty}\frac{B_{2k}\cos\{(2k-1)\beta\}}{k(2k-1)r^{2k-1}}.$$

Proof. Using (64) and Entry 23, we find that, as $x \to \infty$,

$$\log\left\{2\pi(x^2+n^2)^{n-1/2}\prod_{k=0}^{\infty}\left(1+\frac{x^2}{(n+k)^2}\right)\right\}$$

$$= \log\left\{\frac{2\pi\{(n-1)!\}^2(x^2+n^2)^{n+1/2}}{\Gamma(n+1+ix)\Gamma(n+1-ix)}\right\}$$

$$= \log(2\pi)+2\log\Gamma(n)+(2n+1)\log r-\log\Gamma(n+1+ix)-\log\Gamma(n+1-ix)$$

$$\sim \log(2\pi)+2\log\Gamma(n)+(2n+1)\log r-(n+ix+1/2)\log(n+ix)$$

$$-(n-ix+1/2)\log(n-ix)-\log(2\pi)+2n$$

$$-\sum_{k=1}^{\infty}\frac{B_{2k}\{(n+ix)^{2k-1}+(n-ix)^{2k-1}\}}{2k(2k-1)r^{2(2k-1)}}.$$

The desired result now follows since $n+ix = re^{i\beta}$.

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