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ON ADDITIVE PARTITIONS OF SETS OF POSITIVE INTEGERS

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Consider any set $U=\{u_n\}$ with elements defined by $u_{n+2}=u_{n+1}+u_n$, $n\geq 1$, where u_1 and u_2 are relatively prime positive integers. We show that if $u_1<u_2$ or $2\mid u_1u_2$, then the set of positive integers can be partitioned uniquely into two disjoint sets such that the sum of any two distinct members of any one set is never in U. If $u_1>u_2$ and $2\nmid u_1u_2$, no such partition is possible. Further related results are proved which generalize theorems of Alladi, Erdös, and Hoggatt.

1. Introduction and notation

Let $\mathbb{N}$ be the set of positive integers and let $U, S\subset\mathbb{N}$. We say *U splits S* if there exist disjoint sets A and B with $S=A\cup B$ such that $c+d\notin U$ whenever c and d are distinct elements both in A or both in B. We call $A\cup B$ a U-partition of S.

Consider from now on a fixed set $U=\{u_n\}$ with elements defined by $u_{n+2}=u_{n+1}+u_n$, $n\geq 1$. Fix the notation $a=u_1$, $b=u_2$, $x=\frac{1}{2}(a-b)$, $y=\frac{1}{2}(a+b)$, $z=\frac{1}{2}(a+3b)$, and assume throughout that $(a,b)=1$. In [1, Section 2], Alladi, Erdös, and Hoggatt proved that U splits $\mathbb{N}$ uniquely when $a=1$, and they gave examples [1, p. 206] to show that U need not split $\mathbb{N}$ when $a\neq 1$. In this paper, we prove more generally (see Corollary 4) that U splits $\mathbb{N}$ uniquely except when $a>b$, $2\nmid ab$, in which case U fails to split $\mathbb{N}$. We also prove the general result (Theorem 3) that U splits S if and only if not all of x, y, z are in S.

Fix the notation $f_n(r)=r-u_n[r/u_n]$. Thus $f_n(r)$ is the least nonnegative residue of $r \pmod{u_n}$. In Theorem 2, we characterize the elements of the sets in a U-partition of $\mathbb{N}$, in terms of the values of the functions f_n, $n\geq 2$. In Theorem 1, we exhibit a class of sets S for which any U-partition of $\{s\in S\colon s<\max(a,b)\}$ can be uniquely extended to a U-partition of S. We remark that Theorem 1 can be easily extended to yield similar results with U replaced by U', where $U'=\{u'_n\}$ is defined for any fixed $k>2$ by $u'_{n+k}=u'_{n+k-1}+\cdots+u'_n$.

In Section 3, we prove a result (Theorem 7) which generalizes that of [1, Theorem 3.6]. This result shows in particular that if $a<b<m$, $m\notin U$, then m is the sum of two distinct elements both in L or both in R where $L\cup R$ is a U-partition of $\mathbb{N}$. Also in Section 3, we answer the following question posed in [1, p. 211]: Does a saturated set split $\mathbb{N}$ uniquely?

2. Unique U-partitions

Theorem 1. *Suppose that $u-s\in S$ whenever $s\in S$ and u is the smallest element of U exceeding s. Suppose further that not all of x,y,z are in S. Then given $m\in S$ with $m\geq\max(a,b)$, any U-partition of $\{s\in S: s<m\}$ can be uniquely extended to a U-partition of S.*

Proof. Let $A\cup B$ be a U-partition of $\{s\in S: s<m\}$. It suffices to show that m can be adjoined to exactly one of A, B to yield a U-partition of $\{s\in S: s\leq m\}$.

For some $n\geq 2$, $u_n\leq m<u_{n+1}$. Let $q=u_{n+1}-m$. Since $m\geq\max(a,b)$, $q<m$. By the initial hypothesis of Theorem 1, $q\in S$. Thus $q\in A\cup B$; say $q\in A$. Since $q+m\in U$, m cannot be adjoined to A. Suppose for the purpose of contradiction that m cannot be adjoined to B. Then $t+m=u_k$ for some $t\in B$, $k\in\mathbb{N}$. Since $q\in A$, $q\neq t$; thus $k\geq n+1$. If $k\geq n+3$, we would have the contradiction

$$2m>m+t\geq u_{n+3}=u_{n+2}+u_{n+1}>2u_{n+1}.$$

Thus $k=n+2$ and $t+m=u_{n+2}$. Since $m+q=u_{n+1}$, we have $t-q=u_n$.

Assume that $2t=u_{n+1}$. Then $n=2$, for if $n>2$, we would have $2t=2(u_n+q)>2u_n>u_{n+1}$. Thus $q=x$, $t=y$, and $m=z$, which contradicts the hypothesis that not all of x, y, z are in S. Therefore, $2t\neq u_{n+1}$.

Let $v=u_{n+1}-t$. Note that $v<m$. By the initial hypothesis of Theorem 1, $v\in S$. Therefore, $v\in A\cup B$. Since $t\in B$, $v\neq t$, and $v+t\in U$, we have $v\in A$. Since $m\neq t$, $v\neq q$. Thus v and q are distinct elements of A whose sum is $u_{n-1}\in U$, a contradiction. ☐

Example. Theorem 1 shows that the Fibonacci set $U=\{1,2,3,5,\ldots\}$ uniquely splits both $\mathbb{N}$ and $U\cup 2U$.

Theorem 2. *Suppose that $a<b$. For each $n\geq 2$, there is a unique U-partition $L_n\cup R_n$ of $C_n=\{m\in\mathbb{N}: m<u_n\}$, where $L_n=\{f_n(u_{n-1}j):\ 1\leq j\leq\frac{1}{2}u_n\}$ and $R_n=\{f_n(u_{n-1}j):\frac{1}{2}u_n<j<u_n\}$. Also, there is a unique U-partition $L\cup R$ of $\mathbb{N}$, where $L=\bigcup_{2|n}L_n$ and $R=\bigcup_{2|n}R_n$.*

Proof. We first show that $L_2\cup R_2$ is a U-partition of $C_2=\{1,2,\ldots,b-1\}$. Clearly C_2 equals the disjoint union $L_2\cup R_2$. Suppose that $f_2(aj)+f_2(ak)=u\in U$, where $1\leq j<k\leq\frac{1}{2}b$. Since u must be a, b, or $a+b$, we have $a(j+k)\equiv 0$ or $a(\text{mod } b)$, so $j+k\equiv 0$ or $1(\text{mod } b)$, which is impossible. Thus no two distinct elements of L_2 (and similarly of R_2) can add up to an element of U, so $L_2\cup R_2$ is a U-partition of C_2.

We can now invoke Theorem 1 to see that there is a U-partition $L'_n\cup R'_n$ of C_n for each $n\geq 2$. We have $f_n(u_{n-1})\in L'_n$, say. Since $f_n(u_{n-1})+f_n(-u_{n-1})=u_n\in U$, we have $f_n(-u_{n-1})\in R'_n$ (if $u_n>2$). Since $f_n(2u_{n-1})+f_n(-u_{n-1})\in\{u_{n-1}, u_{n+1}\}\subset U$, we have $f_n(2u_{n-1})\in L'_n$ (if $u_n>3$). Continuing in this manner, we see that $L'_n=L_n$. Thus $L_n\cup R_n$ is the unique U-partition of C_n.

By Theorem 1, there exists a unique U-partition $L' \cup R'$ of $\mathbb{N}$, and $L' \cup R'$ extends each U-partition $L_n \cup R_n$. Since $u_{n-1} \in L_n$, $u_{n-1} \in R_{n+1}$ for each $n \geq 2$, it follows that

$$L_2 \subset R_3 \subset L_4 \subset R_5 \subset L_6 \subset R_7 \subset \cdots. \tag{1}$$

We have $\{u_k\colon 2 \nmid k\} \subset L'$, $\{u_k\colon 2 \mid k\} \subset R'$, say. Thus $L' = L$ and $L \cup R$ is the unique U-partition of $\mathbb{N}$. □

Theorem 3. *U splits S if and only if not all of x, y, z are in S.*

Proof. Suppose that $x, y, z \in S$. Since $x+y, x+z, y+z \in U$, clearly U cannot split S.

Conversely, suppose that not all of x, y, z are in S. If $a < b$, then U splits $\mathbb{N}$ by Theorem 2, so U splits S. Hence assume $b > a$. Let $U_a = U - \{a\} = \{b, a+b, a+2b, \ldots\}$. By Theorem 2, one has a U_a-partition $G_3 \cup H_3$ of $C_3 \cap S = \{m \in S\colon m < a+b\}$ which can be extended to a U_a-partition $G \cup H$ of S, where

$$G \supset G_3 = L_3 \cap S, \qquad H \supset H_3 = R_3 \cap S.$$

We now show that x, y is the only possible pair of distinct elements of $G \cup H$ which can add up to a. Write

$$f_3(bj) + f_3(bk) = a, \quad 1 \leq j < k < a+b.$$

Then $b(j+k) \equiv a \pmod{a+b}$, so $(j+k) \equiv -1 \pmod{a+b}$. Thus $j = y-1$, $k = y$, so ab is odd and $f_3(bj) = x$, $f_3(bk) = y$. It follows that if $x \notin S$ or $y \notin S$, then no two distinct elements of S can add up to a. This proves that the U_a-partition $G \cup H$ is in fact a U-partition of S, when $x \notin S$ or $y \notin S$.

It remains to produce a U-partition of S in the case $x, y \in S$, $z \notin S$. (We note that this case does not occur when $S = \mathbb{N}$, so Corollary 4 below is now proved.) Suppose that $x, y \in S$, $z \notin S$. We may suppose without loss of generality that $S = \mathbb{N} - \{z\}$, since if U splits a set, it splits any subset of it. Let $I_3 = G_3 - \{y\}$, $J_3 = H_3 \cup \{y\}$. We now show that $I_3 \cup J_3$ is a U-partition of $C_3 \cap S$. To do so, it suffices to show that $y + r \notin U$ for $r \in H_3$. Suppose that $y + r \in U$. Since $y + r = \frac{1}{2}(a+b) + r < \frac{3}{2}(a+b) < 2a + 3b = u_5$,

$$y + r \in \{a, a+b, a+2b\}.$$

Thus $r \in \{x, y, z\}$, which is impossible, since $\{x, y, z\}$ is disjoint from H_3. This proves that $I_3 \cup J_3$ is a U-partition of $C_3 \cap S$.

We now show that a number $u_n \pm z$ in $I_3 \cup J_3$ is in I_3 if and only if n is odd. This is true for $n \geq 3$, since $u_3 - z = x \in I_3$ and $u_4 - z = y \in J_3$. If $u_2 + z \in I_3 \cup J_3$, then

$$u_2 + z = \tfrac{1}{2}(a+5b) = f_3(b(\tfrac{1}{2}(a+b)+2)) \in J_3.$$

Finally, if $u_1 - z \in I_3 \cup J_3$, then

$$u_1 - z = \tfrac{1}{2}(a-3b) = f_3(b(\tfrac{1}{2}(a+b)-2)) \in I_3.$$

Fix $m \in S$ with $m \geq a+b$. Assume that $A \cup B$ is any U-partition of $\{s \in S: s < m\}$ with the property that a number $u_n \pm z \in A \cup B$ is in A if and only if n is odd. (In the case $m = a+b$, this holds for $A = I_3$, $B = J_3$.) We will show that m can be adjoined to one of A, B to yield a U-partition of $\{s \in S: s \leq m\}$ and that if $m = u_n \pm z$, then m can be adjoined to A or B according as n is odd or even. This will imply the desired result, that the U-partition $I_3 \cup J_3$ can be extended to a U-partition of S.

For some fixed $n \geq 3$, $u_n \leq m < u_{n+1}$. We will use frequently the fact that if $m + \gamma \in U$ for some $\gamma \in A \cup B$, then $m + \gamma < 2m < 2u_{n+1} < u_{n+3}$, so γ equals $u_{n+1} - m$ or $u_{n+2} - m$.

Case 1. $m = u_{n+1} - z$.

We consider only the case $2 \nmid n$, as the case $2 \mid n$ is similar. Assume that m cannot be adjoined to B. Then $m + \beta \in U$ for some $\beta \in B$, so β equals z or $u_n + z$. Since $z \notin S$ and $u_n + z \notin B$ (as n is odd), this is a contradiction.

Case 2. $m = u_n + z$.

We again consider only the case $2 \nmid n$. Assume that m cannot be adjoined to A. Then $m + \alpha \in U$ for some $\alpha \in A$, so α equals $u_{n-1} - z$ or $u_{n+1} - z$. This is impossible, because n is odd.

Case 3. $m = u_k + z$ with $k < n$.

Since $u_n \leq m < u_{n+1}$, we cannot have both $k = 1$ and $n = 3$. Thus $z = m - u_k \geq u_n - u_k \geq u_{n-2}$. However, $z = \frac{1}{2}(a+3b) < a+b = u_3$. Thus $n \leq 4$.

Suppose first that $n = 3$. Thus $u_k = b$ and $3b > a$. Since $k = 2$, we must show that m can be adjoined to B. Assume that $m + \beta \in U$ for some $\beta \in B$, so β equals $u_3 - z$ or $\frac{1}{2}(3a+b)$. Now, $u_3 - z \notin B$ because 3 is odd, so $\beta = \frac{1}{2}(3a+b)$. Since $3b > a$, $\frac{1}{2}(3b-a) = f_3(b(\frac{1}{2}(a+b)+2)) \in B$. Hence $\frac{1}{2}(3b-a)$ and β are distinct elements of B whose sum is $u_4 \in U$, a contradiction.

Suppose now that $n = 4$. Then $k \in \{1, 3, 4\}$. First consider the case $k = 4$. If $m + \beta \in U$ for some $\beta \in B$, then β equals $u_3 - z$ or $u_5 - z$, which is impossible, because 3 and 5 are odd. Thus m can be adjoined to B. Now consider the case $k \in \{1, 3\}$. Assume that $m + \alpha \in U$ for some $\alpha \in A$ with $\alpha \neq m$. Then α equals $u_5 - u_k - z$ or $u_6 - u_k - z$. In the former case, α equals z or $u_4 - z$, which is impossible because z, $u_4 - z \notin A$. Thus $\alpha = u_6 - u_k - z$. This implies that $\alpha = u_4 + z$ or $\alpha = u_6/2 = m$, which is impossible because $u_4 + z \notin A$ and $\alpha \neq m$.

Case 4. $m + z, m - z \notin U$.

To show that m can be adjoined to one of A, B, we can follow verbatim the proof of Theorem 1, except that we have to justify the assertions $q \in S$, $v \in S$ in a different way, since here the initial hypothesis of Theorem 1 is not valid. To see that $q = u_{n+1} - m$ is in $S = \mathbb{N} - \{z\}$, note that $u_{n+1} - m \neq z$ (in Case 4). To see that $v = u_{n+1} - t$ is in S, assume that $v = z$. Then $u_{n+1} - z = t \in B$ and $u_{n-1} - z = u_{n-1} - v = q \in A$, which is impossible, since $n-1$ and $n+1$ have the same parity. □

Corollary 4. *U splits $\mathbb{N}$ if and only if $a < b$ or $2 \mid ab$. Also, U splits $\mathbb{N}$ uniquely if $a < b$ or $2 \mid ab$.*

Proof. The first assertion follows from Theorem 3, and uniqueness is a consequence of Theorem 2. □

3. Extremal sets partitioning $\mathbb{N}$

Let $a < b$. As in Theorem 2, let $L_n \cup R_n$ and $L \cup R$ be the unique U-partitions of C_n and $\mathbb{N}$, respectively. No element of U can be a sum of two distinct elements both in L or both in R. Theorem 7 below shows however, that any $m \in \mathbb{N} - U$ with $m > b$ is a sum of two distinct elements both in L or both in R. This implies, for example, that no set properly containing the set of Fibonacci numbers can split $\mathbb{N}$. In the case $a = 1$, Theorem 7 reduces to [1, Theorem 3.6].

Lemma 5. *Let $a < b$. Fix $n \geq 3$. Then $2u_{n-1}$ can be uniquely expressed as a sum of distinct elements c, d such that $c, d \in L$ or $c, d \in R$. Moreover, $c, d \in L$, $2u_{n-1} \in R$ or $c, d \in R$, $2u_{n-1} \in L$, according as n is odd or even.*

Proof. Suppose that

$$2u_{n-1} = c + d, \quad \text{with } c \neq d, \text{ and } c, d \in L \text{ or } c, d \in R. \tag{2}$$

Since $2u_{n-1} < u_{n+1}$, $c, d \in L_{n+1} \cup R_{n+1}$. Write

$$c = f_{n+1}(u_n j), \quad d = f_{n+1}(u_n k), \qquad 1 \leq j < k < u_{n+1}. \tag{3}$$

Then $2u_{n-1} \equiv u_n(j+k) \pmod{u_{n+1}}$, so $j + k \equiv -2 \pmod{u_{n+1}}$. It follows that $c, d \in L_{n+1}$ and

$$\begin{cases} j = \frac{1}{2}u_{n+1} - 2, & k = \frac{1}{2}u_{n+1} \quad \text{if } 2 \mid u_{n+1}, \\ j = \frac{1}{2}u_{n+1} - \frac{3}{2}, & k = \frac{1}{2}u_{n+1} - \frac{1}{2} \quad \text{if } 2 \nmid u_{n+1}. \end{cases} \tag{4}$$

This proves that there is at most one pair c, d satisfying (2). Moreover, if c, d are defined by (3) and (4), then

$$\begin{aligned} c &= -\tfrac{1}{2}u_{n+1} + 2u_{n-1}, & d &= \tfrac{1}{2}u_{n+1} & &\text{if } 2 \mid u_{n+1}, \\ c &= \tfrac{3}{2}u_{n-1}, & d &= \tfrac{1}{2}u_{n-1} & &\text{if } 2 \nmid u_{n+1}, 2 \nmid u_n, \\ c &= u_{n+1} - \tfrac{3}{2}u_n, & d &= u_{n+1} - \tfrac{1}{2}u_n & &\text{if } 2 \nmid u_{n+1}, 2 \mid u_n, \end{aligned}$$

so (2) indeed holds. Finally, note that $2u_{n-1} = f_{n+1}(u_n(u_{n+1} - 2)) \in R_{n+1}$, so since $c, d \in L_{n+1}$, the last assertion of Lemma 5 follows from (1). □

Lemma 6. *Let $a < b$. Then $2a$ can be expressed as a sum of distinct elements c, d with $c, d \in L$ or $c, d \in R$, if and only if either*

$$2 \mid a; \quad or \quad 2 \nmid ab, 3a > b; \quad or \quad 2 \mid b, 2a > b. \tag{6}$$

Also, $b - a$ can be expressed as a sum of distinct elements e, f with $e, f \in L$ or $e, f \in R$, if and only if

$$2 \mid b, \quad 2a < b. \tag{7}$$

Proof. The proof of Lemma 5 up through (4) holds for $n = 2$. The values of c in (5) when $n = 2$ are positive if and only if (6) holds, so the first assertion of Lemma 6 holds. An easy similar argument verifies the second assertion of Lemma 6. □

Theorem 7. *Let $a < b$. Let $m \in \mathbb{N}$, $m > a$, $m \notin U \cup \{2a, b-a\}$. Then m is the sum of two distinct elements both in L or both in R. This conclusion is also valid when either $m = 2a$ and (6) holds, or $m = b - a$ and (7) holds.*

Proof. The last assertion follows from Lemma 6. Say $m \notin U \cup \{2a, b-a\}$. If $m \in 2U$, the result follows from Lemma 5, so assume $m \notin 2U$. For some $n \geq 1$, $u_n < m < u_{n+1}$, so $m \in L_{n+1} \cup R_{n+1}$. First suppose $m \in L_{n+1}$. Then $m = f_{n+1}(u_n j)$ with $1 < j \leq \frac{1}{2}u_{n+1}$. Thus $m - u_n = f_{n+1}(u_n(j-1)) \in L_{n+1}$, and since $u_n = f_{n+1}(u_n) \in L_{n+1}$, $m = (m - u_n) + u_n$ is the sum of two distinct elements both in L or both in R. Now suppose $m \in R_{n+1}$. Then $m = f_{n+1}(u_n k)$ with $\frac{1}{2}u_{n+1} < k \leq u_{n+1} - 1$. We cannot have $k = u_{n+1} - 1$, for if $n = 1$, this would imply $m = b - a$, and if $n > 1$, this would imply $m = u_{n-1}$. Thus, $c = f_{n+1}(u_n(k+1)) \in R_{n+1}$. Note that $d = f_{n+1}(u_n(u_{n+1} - 1)) \in R_{n+1}$, and that $d = b - a$ or $d = u_{n-1}$ according as $n = 1$ or $n > 1$. Thus $m = c + d$, so m is the sum of two distinct elements both in L or both in R. □

We conclude this section by giving a negative answer to the following question posed in [1, p. 211]: Does a saturated set split $\mathbb{N}$ uniquely?

(A set V with $\{1, 2\} \subset V \subset \mathbb{N}$ is *saturated* [1, Def. 3.5] if V splits $\mathbb{N}$ but no set of positive integers properly containing V splits $\mathbb{N}$.) We will exhibit a saturated set V which splits $\mathbb{N}$ in two ways.

Let $W = \{1, 2, 3, 4\} \cup \{2^n + 4 : n \geq 2\}$. There is a unique W-partition of $2\mathbb{N} - 1$ (the set of odd positive integers), namely $A_1 \cup A_2$, where $A_1 = 4\mathbb{N} + 1$, $A_2 = 4\mathbb{N} - 1$. There is also a unique W-partition of $4\mathbb{N}$, namely $B_1 \cup B_0$, where $B_1 = 8\mathbb{N} - 4$, $B_0 = 8\mathbb{N}$. Furthermore, there is a unique W-partition $D_1 \cup D_2$ of $4\mathbb{N} - 2$. Say $2 \in D_2$. There are exactly two W-partitions of $\mathbb{N}$, namely $G_i \cup H_i$ $(i = 0, 1)$, where $G_i = (A_1 \cup D_1) \cup B_i$ and $H_i = (A_2 \cup D_2) \cup B_{1-i}$. Let V be the set obtained by adjoining to W every $m \in \mathbb{N}$ possessing the property that for each set

$J \in \{G_0, G_1, H_0, H_1\}$, no two distinct elements of J add up to m. Then there are two V-partitions of $\mathbb{N}$, namely $G_0 \cup H_0$ and $G_1 \cup H_1$.

Suppose for the purpose of contradiction that V is not saturated. Then there exists $m \in \mathbb{N}$ and $i \in \{0, 1\}$ such that $G_i \cup H_i$ is a $(V \cup \{m\})$–partition of $\mathbb{N}$ but $G_{1-i} \cup H_{1-i}$ is not. Thus $m = c + d$ where $c \neq d$ and either $c, d \in G_{1-i}$ or $c, d \in H_{1-i}$. At least one of c, d is a multiple of 4, for otherwise we'd have $c, d \in G_i$ or $c, d \in H_i$. If m is odd, then $m = 4 + (m-4) = 8 + (m-8)$ is the sum of two distinct elements both in G_i or both in H_i, a contradiction. If $2 \| m$, then $m = (\frac{1}{2}m - 2) + (\frac{1}{2}m + 2)$ is the sum of two distinct elements both in G_i or both in H_i. Thus m,c, and d are multiples of 4. Therefore $c, d \in B_1$ or $c, d \in B_0$, so $m = c + d$ is the sum of two distinct elements both in G_i or both in H_i. This completes the proof that V is a saturated set which splits $\mathbb{N}$ in two ways.

Reference

[1] K. Alladi, P. Erdös and V. Hoggatt, Jr., On additive partitions of integers, Discrete Math. 22 (1978) 201–211.