Math 264B Assignment 1 Due Feb. 3, 2017 Remmel

1) Let $\theta_n : \mathcal{S}_n \rightarrow \mathcal{S}_n$ be the bijection given in class such that for all $\sigma \in S_n$, $inv(\sigma) = maj(\theta_n(\sigma))$.

(a) Find $\theta_9(482163597)$.

(b) Find $\theta_9^{-1}(764913285)$.

(2) Let $\phi : R(0^k, 1^{n-k}) \rightarrow R(0^k, 1^{n-k})$ be the map defined in Theorem 1.5 of the Mendes-Remmel book such that $maj(r) = inv(\phi(r))$ for all $r \in R(0^k, 1^{n-k})$.

(a) Find $\phi(0110110)$.

(b) Find $\phi^{-1}(1001011)$.

(3) Let

$$\begin{aligned} [n]_{p,q} &= p^{n-1} + p^{n-2}q + \cdots + pq^{n-2} + q^{n-1} = \frac{p^n - q^n}{p-q}, \\ [n]_{p,q}! &= [n]_{p,q}[n-1]_{p,q} \cdots [1]_{p,q}, \text{ and} \\ \begin{bmatrix} n \\ k \end{bmatrix}_{p,q} &= \frac{[n]_{p,q}!}{[k]_{p,q}![n-k]_{p,q}!}. \end{aligned}$$

For any sequence of natural numbers $\sigma = \sigma_1 \ldots \sigma_n$, let

$$inv(\sigma) = \sum_{1 \leq i < j \leq n} \chi(\sigma_i > \sigma_j) \text{ and } coinv(\sigma) = \sum_{1 \leq i < j \leq n} \chi(\sigma_i < \sigma_j).$$

(a) Prove $[n]_{p,q}! = \sum_{\sigma \in S_n} q^{inv(\sigma)} p^{coinv(\sigma)}$.

(b) Prove $\begin{bmatrix} n \\ k \end{bmatrix}_{p,q} = \sum_{r \in \mathcal{R}(1^k 0^{n-k})} q^{inv(r)} p^{coinv(r)}$ where $\mathcal{R}(1^k 0^{n-k})$ is the set of all rearrangements of k 1's and $n-k$ 0's.

(4) Give combinatorial proofs of the following indentities.

(a) $\begin{bmatrix} n \\ k \end{bmatrix}_{p,q} = p^k \begin{bmatrix} n-1 \\ k \end{bmatrix}_{p,q} + q^{n-k} \begin{bmatrix} n-1 \\ k-1 \end{bmatrix}_{p,q}$.

(b) $\begin{bmatrix} n \\ k \end{bmatrix}_{p,q} = q^k \begin{bmatrix} n-1 \\ k \end{bmatrix}_{p,q} + p^{n-k} \begin{bmatrix} n-1 \\ k-1 \end{bmatrix}_{p,q}$.

(c) $\displaystyle \binom{x+n+1}{n} = \sum_{i=0}^{n} \binom{x+i}{i}$.

(d) $\displaystyle \begin{bmatrix} n+1 \\ k \end{bmatrix}_q = \sum_{j=0}^{k} q^{k-j} \begin{bmatrix} n-j \\ k-j \end{bmatrix}_q$.

(5)
(a) By enumerating partitions with respect to the number of parts, the size, and the size of the largest part, show directly that

$$\sum_{m\geq 1} \frac{q^m z w^m}{(1-zq)(1-zq^2)\cdots(1-zq^m)} =$$
$$\sum_{m\geq 1} \frac{q^{m^2} z^m w^m}{(1-zq)(1-zq^2)\cdots(1-zq^m)(1-wq)(1-wq^2)\cdots(1-wq^m)}.$$

(b) By enumerating partitions with respect to the number of parts, the size, and the size of the largest part, show directly that

$$\sum_{m\geq 1} q^m z w^m (1+zq)(1-zq^2)\cdots(1+zq^{m-1}) =$$
$$\sum_{m\geq 1} \frac{q^{\binom{m+1}{2}} z^m w^m}{(1-wq)(1-wq^2)\cdots(1-wq^m)}.$$