# Solution to Problem 0

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**Problem 0.**

We define a class of posets $\{P_n\}$ as follows.

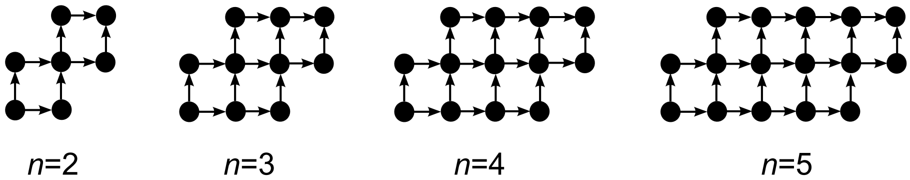


Figure 1: $P_2, P_3, P_4$ and $P_5$

$a_n$ is the number of linear extensions of poset $P_n$. One can figure out what the Hasse diagram of $P_n$ is by observing posets in Figure 1.

Find $a(n)$.

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**Solution.**

Using hook-length formula for shifted standard Young tableaux, we obtain the formula of $a(n)$

$$a(n) = \sum_{k=0}^{n-1} \frac{(2n+k+1)!(n-k+1)(n-k)}{(n+1)!n!k!(2n+1)(n+k+1)(n+k)}.$$

The formula of $a(n)$ is the same as the formula of sequence A181197 on OEIS. For $n \geq 2$, the first few terms are $4, 29, 290, 3532, 49100, \ldots$.

Now we take $P_4$ as an example to demonstrate the method of finding $a(n)$. It is easy to see $P_4$ is equivalent to the poset in Figure 2.

An element in the poset is marked in blue, denoted by $b$. Clearly, $10 \leq b \leq 13$. There are four

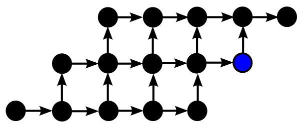

Figure 2: a poset equivalent to $P_4$

choices of $b$ in total so we split the discussion of $b$ into 4 cases.

Case 1: $b = 10$.
In this case, the elements in the top level must be $\{11, 12, 13, 14, 15\}$. Then the poset can be drawn as the poset in Figure 3 which can be simplified to a shifted Young tableau on the right hand side.
Using the hook-length formula for the shifted Young tableaux, we can get the number of linear

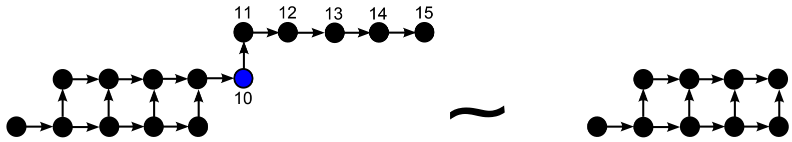


Figure 3: $b = 10$

extensions of the RHS poset

$$\frac{(2n+1)!(n+1)n}{(n+1)!n!0!(2n+1)(n+1)n}, \qquad n = 4.$$

Case 2: $b = 11$.
In this case, the rightmost four elements in the top level must be $\{12, 13, 14, 15\}$. Then the poset can be drawn as the poset in Figure 4 which can be simplified to a shifted Young tableau on the right hand side.
Using the hook-length formula for the shifted Young tableaux, we can get the number of linear

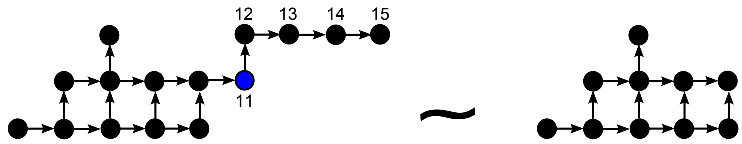


Figure 4: $b = 11$

extensions of the RHS poset

$$\frac{(2n+2)!n(n-1)}{(n+1)!n!1!(2n+1)(n+2)(n+1)}, \qquad n = 4.$$

Case 3: $b = 12$.
In this case, the rightmost three elements in the top level must be $\{13, 14, 15\}$. Then the poset can be drawn as the poset in Figure 5 which can be simplified to a shifted Young tableau on the right hand side.
Using the hook-length formula for the shifted Young tableaux, we can get the number of linear

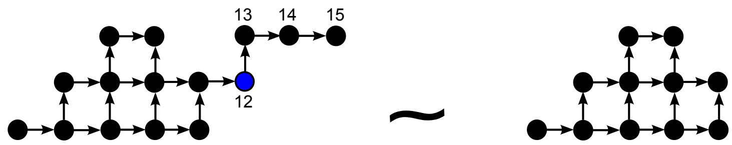


Figure 5: $b = 12$

extensions of the RHS poset

$$\frac{(2n+3)!(n-1)(n-2)}{(n+1)!n!2!(2n+1)(n+3)(n+2)}, \qquad n=4.$$

Case 4: $b=13$.
In this case, the rightmost two elements in the top level must be $\{14,15\}$. Then the poset can be drawn as the poset in Figure 6 which can be simplified to a shifted Young tableau on the right hand side.
Using the hook-length formula for the shifted Young tableaux, we can get the number of linear

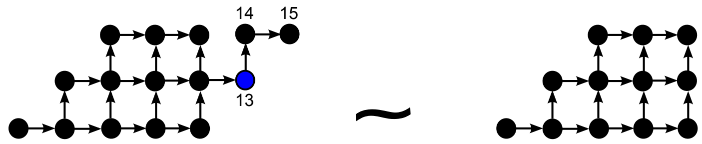


Figure 6: $b=13$

extensions of the RHS poset

$$\frac{(2n+4)!(n-2)(n-3)}{(n+1)!n!3!(2n+1)(n+4)(n+3)}, \qquad n=4.$$

Therefore, we have the formula of $a(4)$ as follows

$$a(4)=\sum_{k=0}^{3}\frac{(2n+k+1)!(n-k+1)(n-k)}{(n+1)!n!k!(2n+1)(n+k+1)(n+k)}, \qquad n=4.$$