Problems and Conjectures

in the

Combinatorial Theory

of Macdonald Polynomials

WHAT IS COMBINATORICS?

Answer:

The study of relations between combinatorial structures

and the resulting interactions with other mathematical disciplines

What are combinatorial structures?

Answer:

Visual representations of mathematical constructs

GOOGLING "Visual representations" gives

Visual representations are a powerful way for students to access abstract math ideas. Drawing a situation, graphing lists of data, or placing numbers on a number line all help to make abstract concepts more concrete, whether done online or offline.

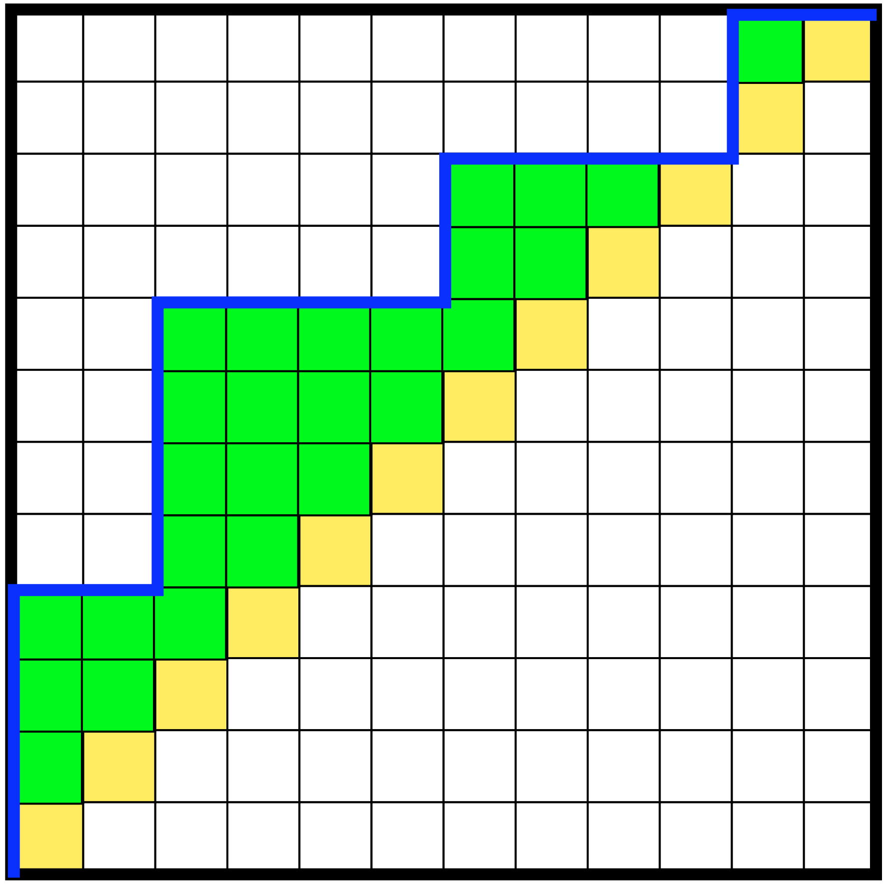
What is a Parking Function?

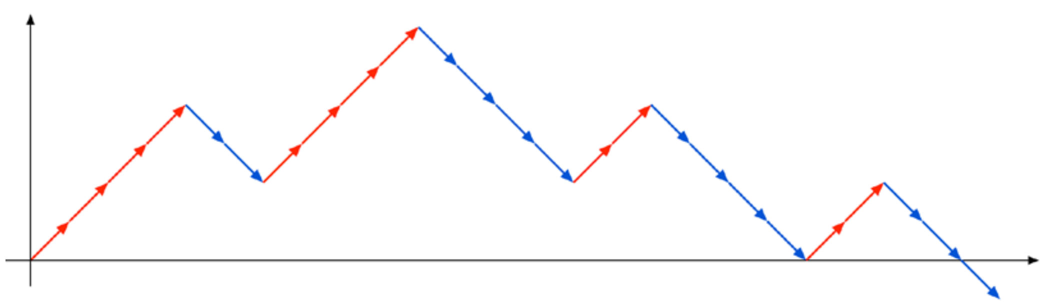
RICHARD STANLEY'S Definition

Easy: Let $\alpha = (a_1, \ldots, a_n) \in \mathbb{P}^n$ Let $b_1 \leq b_2 \leq \cdots \leq b_n$ be the increasing rearrangement of α. Then α is a parking function if and only $b_i \leq i$.

This is a mathematical construct

A Dyck Path and its area





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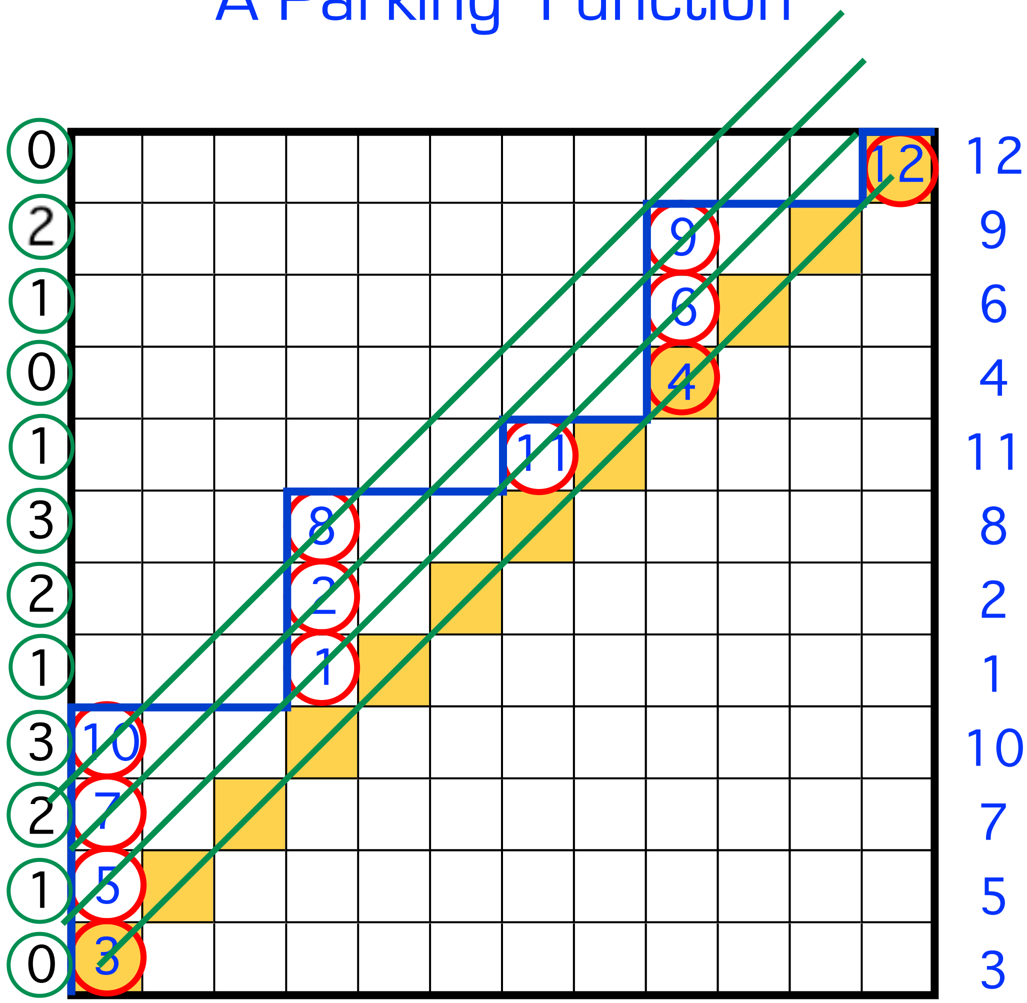
A Parking Function

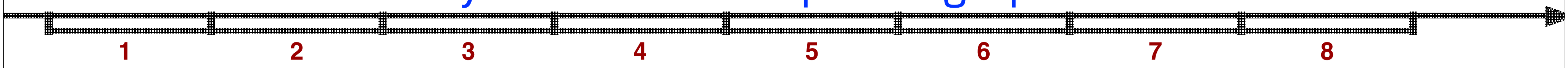


A two line representation used for computer programming purposes

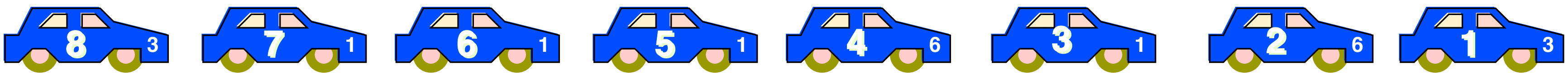
3	5	7	10	1	2	8	11	4	6	9	12
0	1	2	3	1	2	3	1	0	1	2	0

HOW DID WE GET TO THIS REPRESENTATION?

A one way street with 8 parking spaces



eight cars wish to park in it



The large number is their order of arrival

The small number is their preferred parking place

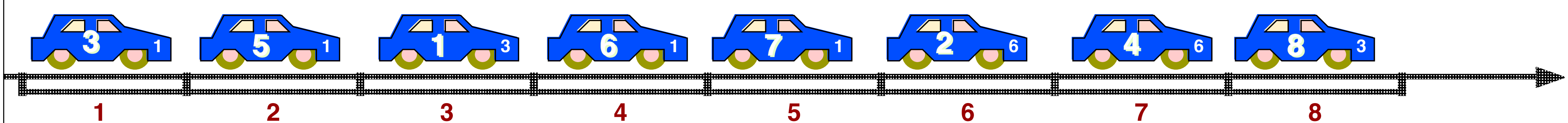
The corresponding Preference Function

x	f(x)
1	3
2	6
3	1
4	6
5	1
6	1
7	1
8	3

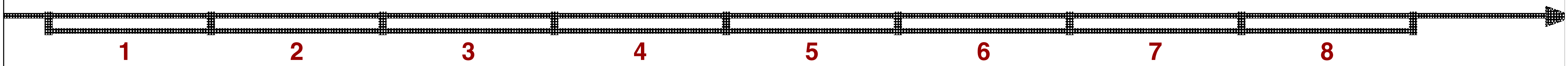
Each car proceeds to its preferred spot

if occupied it parks in the next available spot

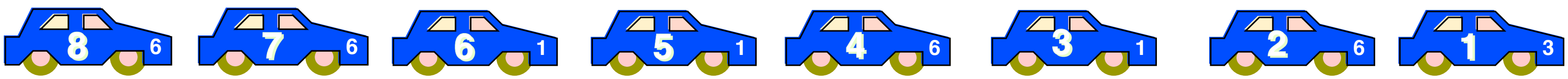
This preference function Parked all the cars



A one way street with 8 parking spaces



eight cars wish to park in it



x	f(x)
1	3
2	6
3	1
4	6
5	1
6	1
7	6
8	6

The large number is their order of arrival

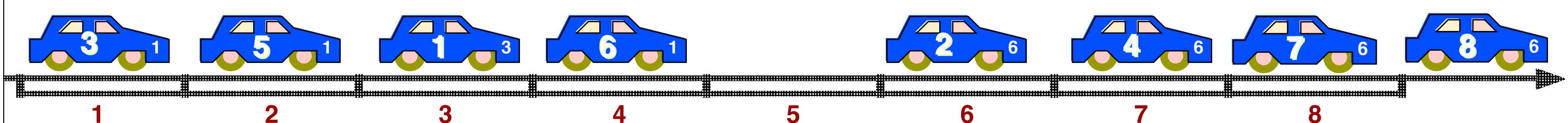
The small number is their preferred parking place

The corresponding Preference Function ⇨

Each car proceeds to its preferred spot

if occupied it parks in the next available spot

This preference function did not park all the cars!



WHY?

the number of cars that want to park in the first 5 places is less than 5 . . .

next

A Parking function is a preference function that parks the cars

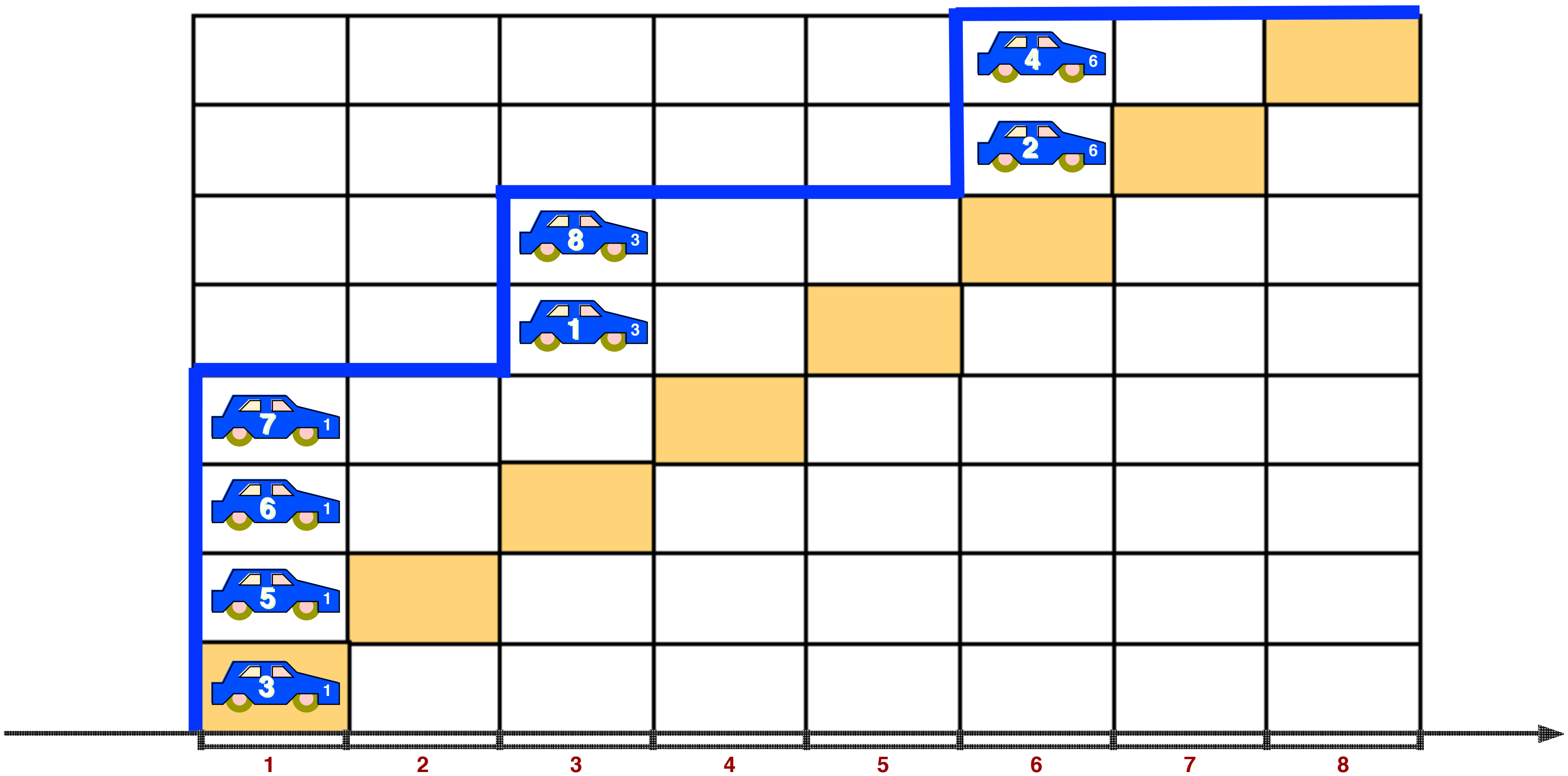
THEOREM

A preference function is a Parking Function

if and only if the number of cars that want to park in the first k places

is greater or equal to k

Parking functions and Dyck Paths



All preference functions constructed in this manner satisfy the necessary condition!

Remarkably this condition is also sufficient!!

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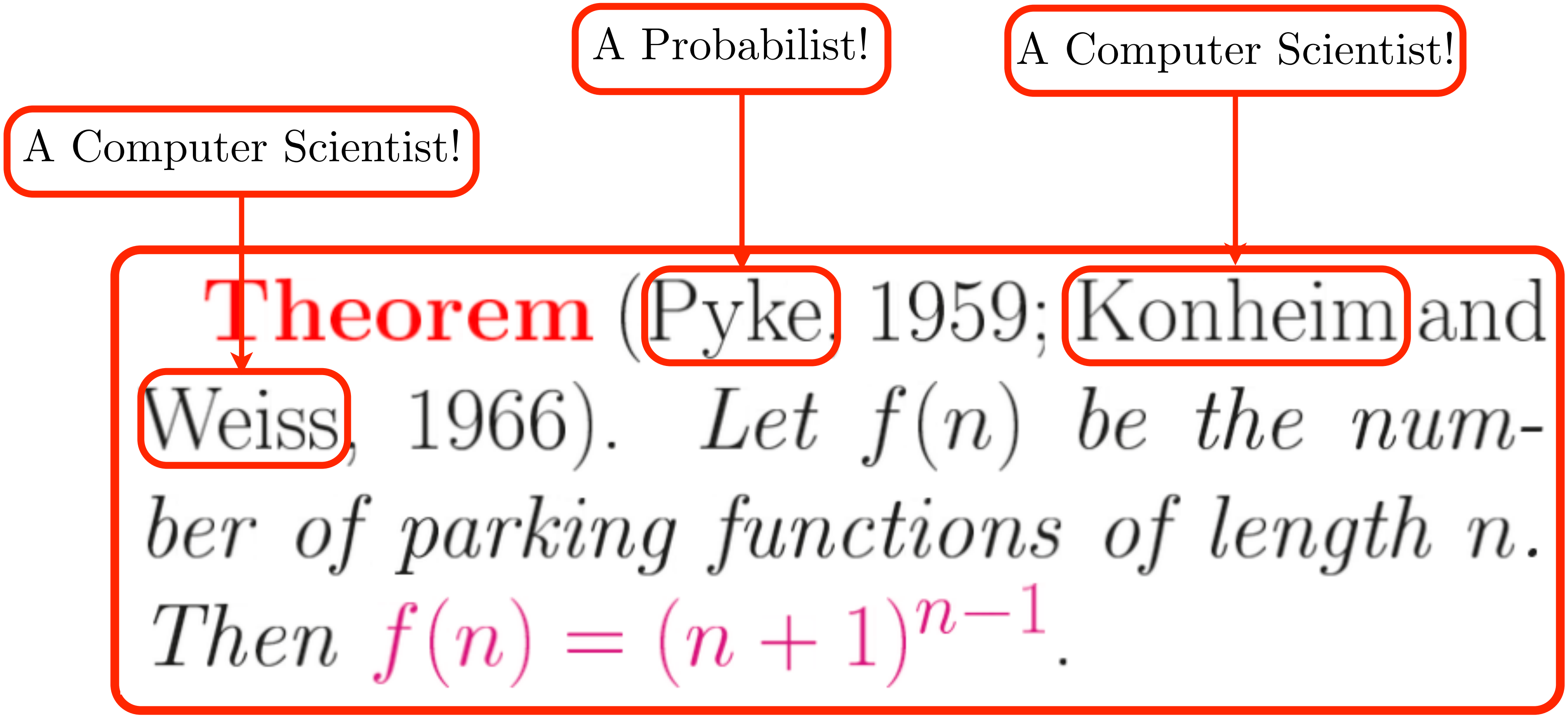
From an online lecture of Richard Stanley



Theorem (Pyke, 1959; Konheim and Weiss, 1966). *Let $f(n)$ be the number of parking functions of length n. Then* $f(n) = (n+1)^{n-1}$.

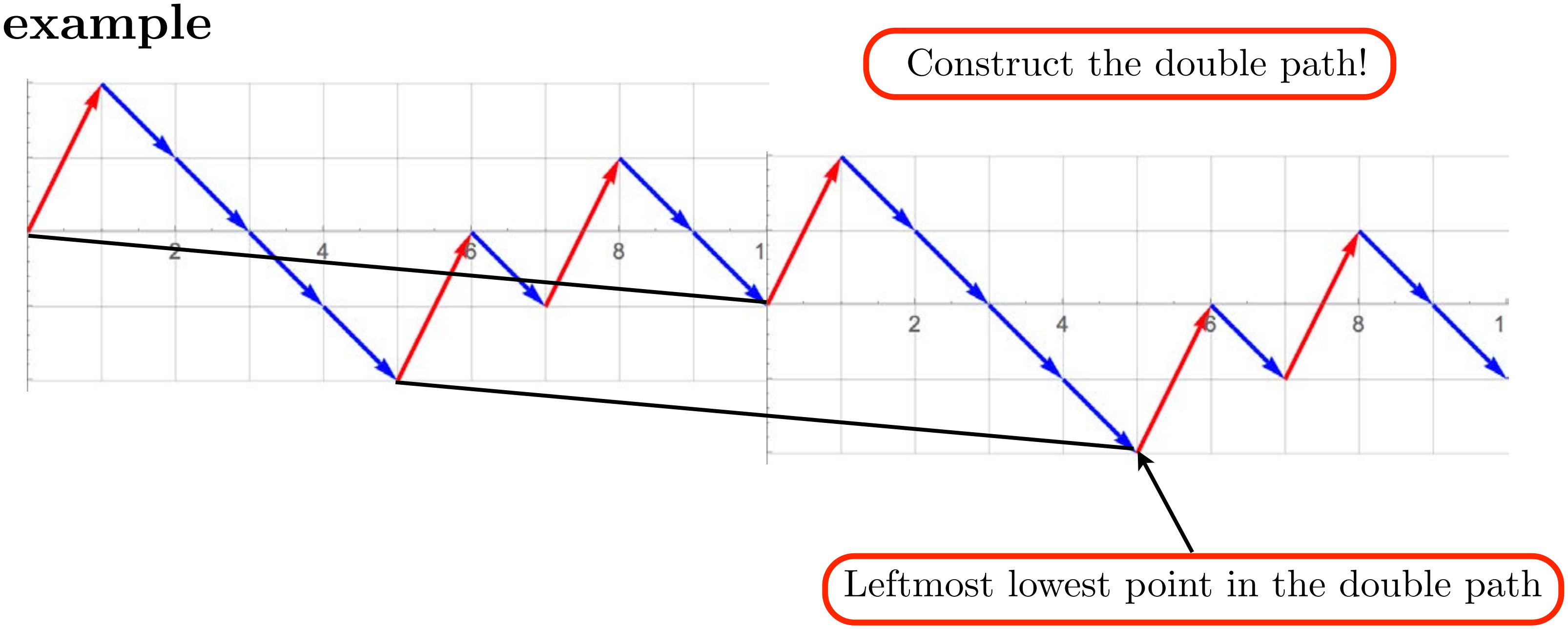
A result in Enumerative Combinatorics

A POWERFUL TOOL IN ENUMERATION

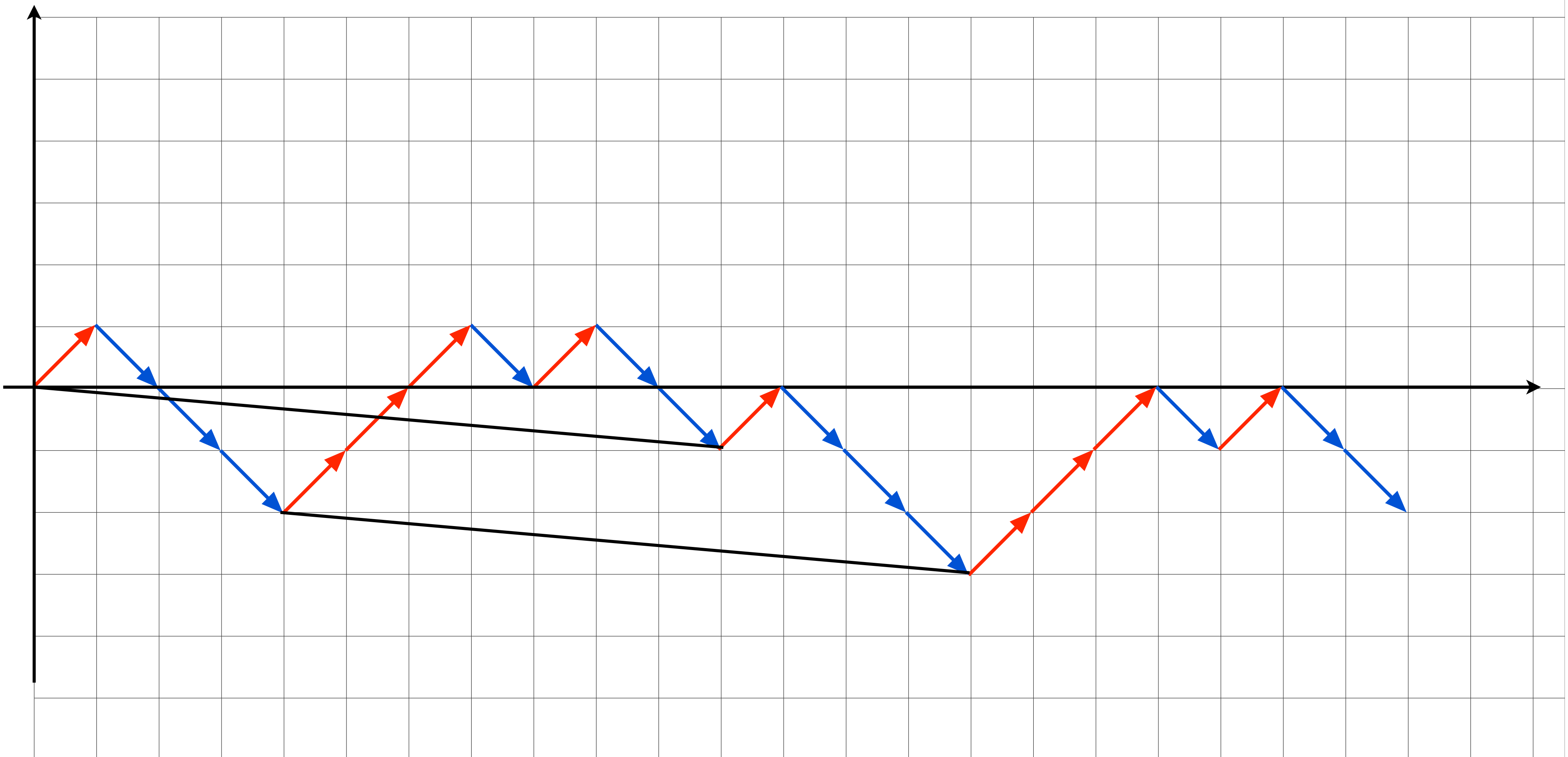
THE CYCLIC LEMMA

- *Let* $\mathcal{M}$ *be an ordered multiset of lattice vectors in the plane*
- *Suppose that* $\sum_{V \in \widetilde{\mathcal{M}}} V$ *is never parallel to* $\sum_{V \in \mathcal{M}} V$ *for any* $\widetilde{\mathcal{M}} \subset \mathcal{M}$
- *Call the segment* $(0,0) \leftrightarrow \sum_{V \in \mathcal{M}} V$ *the "cord" of* $\mathcal{M}$
- *Then there is one and only one cyclic rearrangement of* $\mathcal{M}$ *whose partial sum polygon has vertices all above the cord*

Proof by example



Counting Dyck Paths



The number of paths with n red steps and $n+1$ blue steps is $\binom{2n+1}{n}$

Since there is only one Dyck path in each circular rearrangement class

The number of Dyck paths with n red steps is $\dfrac{1}{2n+1}\binom{2n+1}{n} = \dfrac{1}{n+1}\binom{2n}{n}$

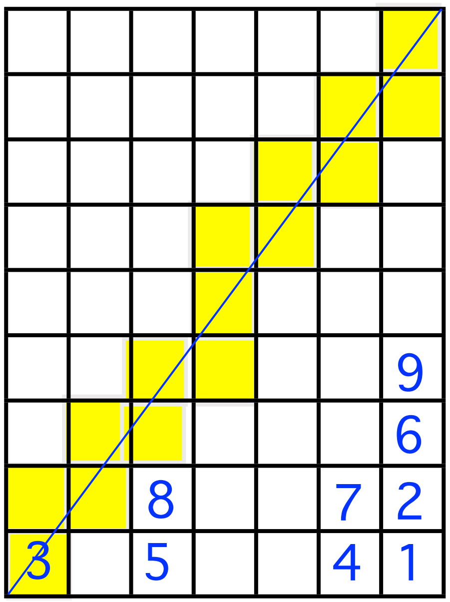
Counting Parking Functions

Theorem

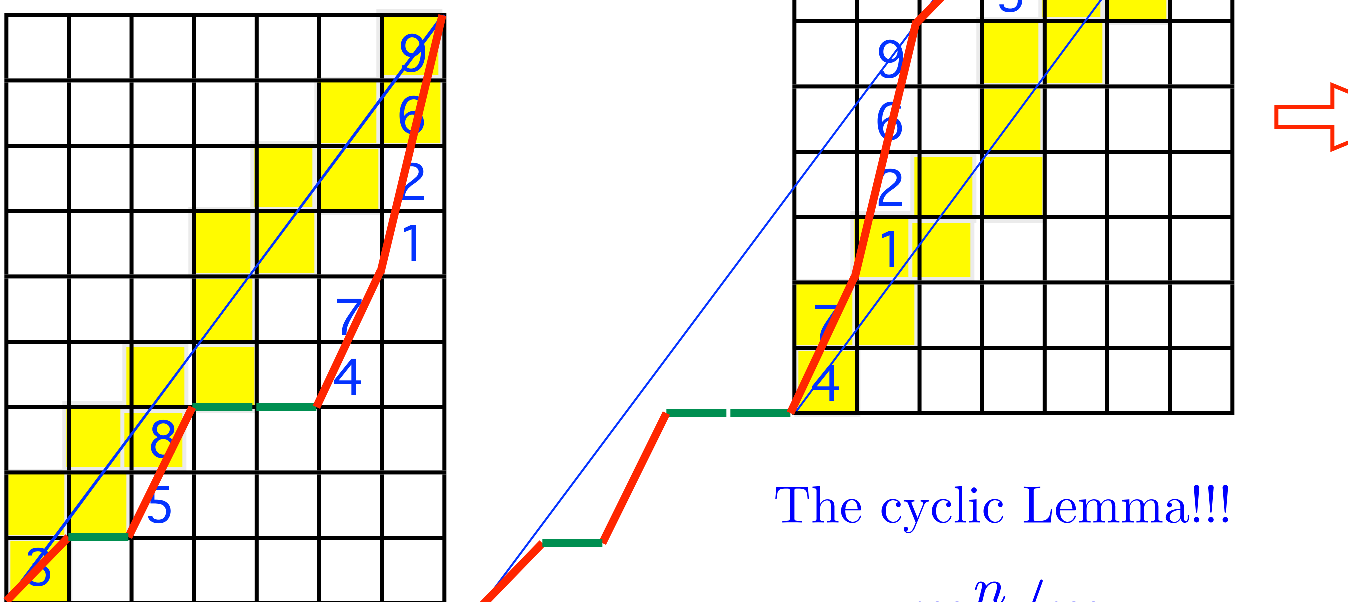
The number of m, n-Parking Functions is

$$\mathbf{m^{n-1}}$$

Proof

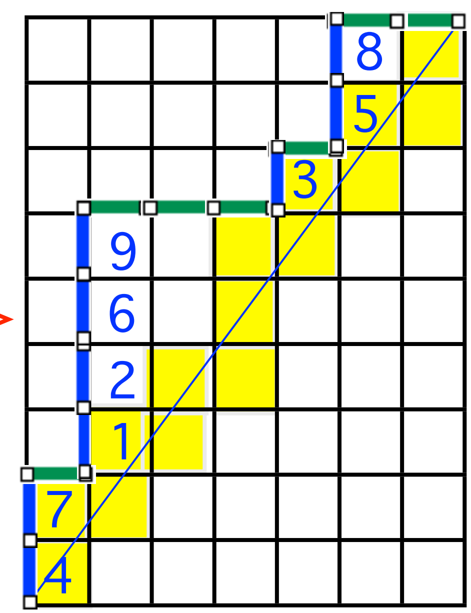


$\longrightarrow \quad m^n$



The cyclic Lemma!!!

$$m^n/m$$



Q.E.D.!!!

m,n Rectangles

Given two relatively prime integers m, n

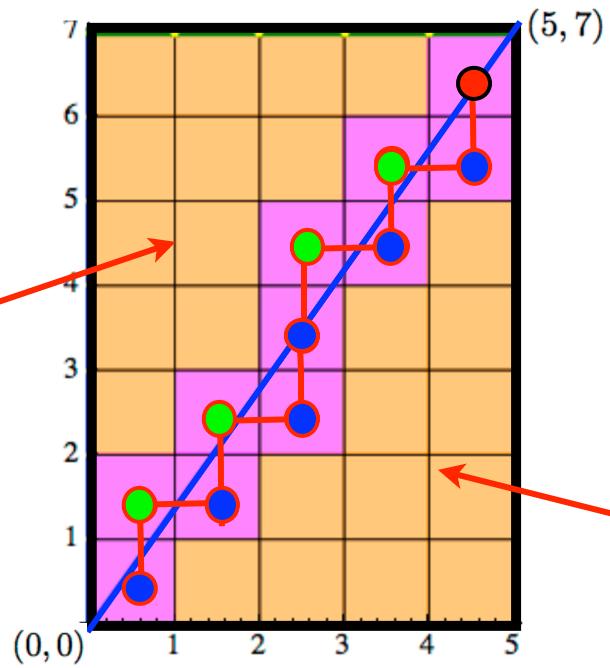
the corresponding "m, n-Rectange" has n rows and m columns.

The line joining $(0,0)$ to (m,n) will be called the "main diagonal"

As we see in the example below for $m = 5$ and $n = 7$

the number of cells above the lattice diagonal is

$$\frac{(m-1)(n-1)}{2}$$



the number of cells below the lattice diagonal is

$$\frac{(m-1)(n-1)}{2}$$

the main diagonal splits (non trivially) each lattice cell that it touches

it is easy to show that the number of split cells is $m + n - 1$

the touched cells (purple) form what we call the "lattice Diagonal"

m,n “Dyck” paths

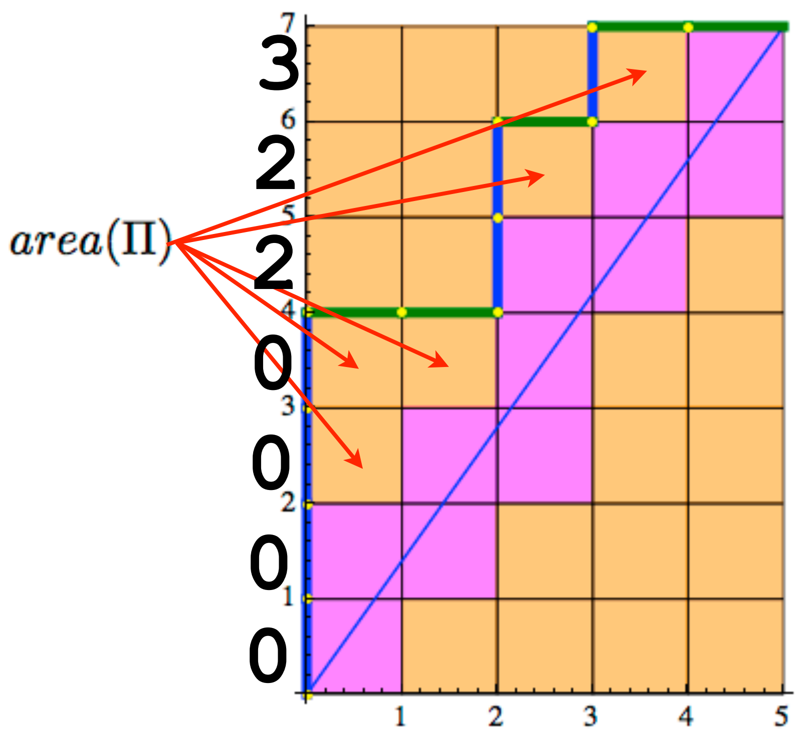
m,n-Dyck paths go from $(0,0)$ to (m,n) by NORTH and EAST steps

always remaining weakly above the lattice diagonal

They are represented on the computer by vectors $U = (u_1, u_2, \ldots, u_n)$

whose coordinates give the horizontal distances of the NORTH steps

from the left side of the rectangle



these U vectors are characterized

by the following requirements

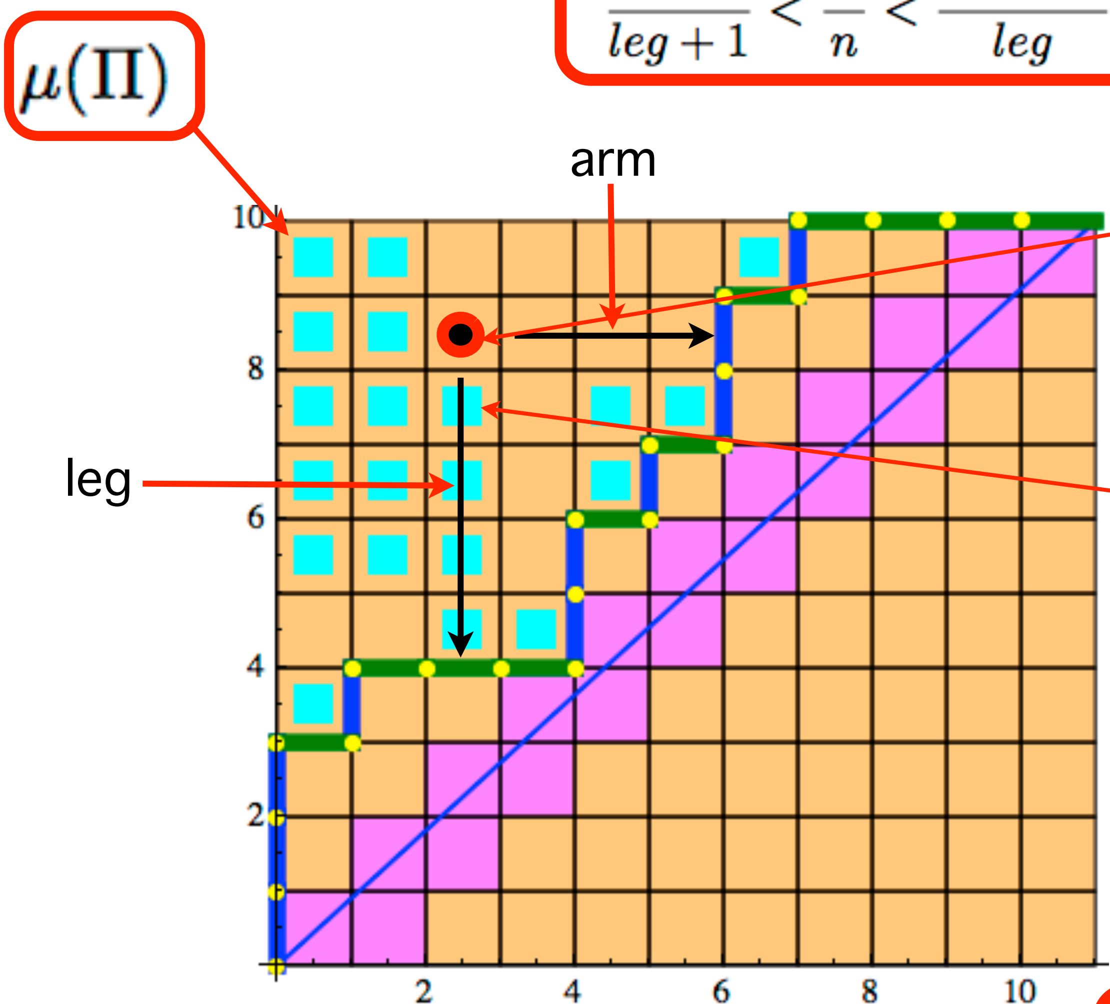
$$u_1 = 0$$
$$u_{i-1} \le u_i \le \frac{m}{n}(i-1)$$

$$area(\Pi) = \frac{(m-1)(n-1)}{2} - u_1 - u_2 - \cdots - u_n$$

The "dinv" of a path

The dinv of an m, n-path is given by the number of cells above the path whose ***arm*** and ***leg*** satisfy the following inequalities

$$\frac{arm}{leg+1} < \frac{m}{n} < \frac{arm+1}{leg}$$



$$\frac{3}{5} < \frac{11}{10} < \frac{4}{4}$$ *no!*

$$\frac{3}{4} < \frac{11}{10} < \frac{4}{3}$$ *yes!*

$$dinv(\Pi) = 20$$

In Summary

Denoting by $\mu(\Pi)$ the Ferrers diagram above Π we set

$$dinv(\Pi) = \sum_{c \in \mu(\Pi)} \chi\left(\frac{a(c)}{l(c)+1} < \frac{m}{n} < \frac{a(c)+1}{l(c)}\right)$$