MATH 287A Winter 2024 Final exam —due Friday March 22 by noon.
Please submit your answers on GRADESCOPE—use code: NPJKJ6

You may use your textbooks, notes and calculator but **do not collaborate with anybody on this exam.** *The first 11 problems have equal weight; the 12th problem has double weight.*

1. Let X_t be a mean zero, (weakly) stationary time series, with spectral density $f(\lambda)$ that is positive for all λ. Define the *innovations* $W_t = X_t - \hat{X}_t$ where $\hat{X}_t$ is the projection of X_t on $\bar{sp}\{X_s, s < t\}$. Show that W_t is a white noise, and give an expression for its variance.

2. Let $X_t = Y_t + W_t$ where the Y series is independent of the W series. Assume Y_t satisfies an AR(1) model (with respect to some white noise), and W_t satisfies a different AR(1) model (with respect to some other white noise). Show that X_t is not AR(1) but it is ARMA(p,q) and identify p and q. [Hint: show that the spectral density of X_t is of the form of an ARMA(p,q).]

3. Let Y_t be a mean zero, (weakly) stationary time series, with lag–k autocovariance $\gamma(k)$. The **inverse autocovariance** $\xi(k), k \in \mathbf{Z}$ is defined as the double sequence satisfying

$$\sum_{k\in Z} \gamma(k)\xi(j-k) = \begin{cases} 1 & \text{if } j = 0 \\ 0 & \text{else.} \end{cases}$$

Consider the problem of optimal linear interpolation in which we have observed all the Y_t except Y_0. Show that $P_{\bar{sp}(Y_t, t\neq 0)} Y_0 = \sum_{k=1}^{\infty} a_k (Y_k + Y_{-k})$, where $a_k = -\xi(k)/\xi(0)$.

[HINT: verify the normal equations using the definition of inverse acf.]

4. Let Y_t satisfy a mean zero, causal AR(p). The claim is that the inverse autocovariance $\xi(k)$ vanishes for $k > p$. Verify the claim in the case of an AR(1) model. [HINT: use problem 3.]

5. Let Y_t satisfy a mean zero, causal AR(p) model with innovation variance σ^2. Denote the autocovariance $\gamma(k)$ and inverse autocovariance $\xi(k)$. Consider the $n \times n$ Toeplitz matrices Γ_n and Ξ_n with ij element $\gamma(i-j)$ and $\xi(i-j)$ respectively. We want to investigate the claim that $\Xi_n \approx \Gamma_n^{-1}$ for large n. In particular, we want to verify the claim that

$$\Gamma_n^{-1} = \Xi_n - \begin{bmatrix} A_p & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & B_p \end{bmatrix} \tag{1}$$

where A_p, B_p are two $p \times p$ symmetric matrices, and $\mathbf{0}$ denotes a matrix of zeros (with appropriate dimension). Verify eq. (1) in the special case of an AR(1) model, i.e., $p = 1$.

HINT: Let $|c| < 1$ and consider the matrix identity:

$$\begin{bmatrix} 1 & c & c^2 & \dots & c^{n-2} & c^{n-1} \\ c & 1 & c & \dots & c^{n-3} & c^{n-2} \\ c^2 & c & 1 & \dots & c^{n-4} & c^{n-3} \\ & \vdots & & \ddots & & \vdots \\ c^{n-2} & c^{n-3} & c^{n-4} & \dots & 1 & c \\ c^{n-1} & c^{n-2} & c^{n-3} & \dots & c & 1 \end{bmatrix}^{-1} = \frac{1}{1-c^2} \begin{bmatrix} 1 & -c & 0 & \dots & 0 & 0 & 0 \\ -c & 1+c^2 & -c & \dots & 0 & 0 & 0 \\ 0 & -c & 1+c^2 & \dots & 0 & 0 & 0 \\ & \vdots & & \ddots & & \vdots & \\ 0 & 0 & 0 & \dots & 1+c^2 & -c & 0 \\ 0 & 0 & 0 & \dots & -c & 1+c^2 & -c \\ 0 & 0 & 0 & \dots & 0 & -c & 1 \end{bmatrix}$$

6. Denote $\hat{\mu}_{blue}$ the BLUE (best linear unbiased estimator) of $\mu = EX_t$ based on data $X_1, \dots, X_n$ satisfying the AR(1) model: $X_t - \mu = \rho(X_{t-1} - \mu) + Z_t$ where $Z_t \sim iid\ (0, \sigma^2)$ and $|\rho| < 1$. Use the above matrix identity to compute $\hat{\mu}_{blue}$ under an AR(1) model, and show that $Var(\bar{X})/Var(\hat{\mu}_{blue}) \to 1$ as $n \to \infty$, i.e., the sample mean $\bar{X} = n^{-1}\sum_{t=1}^{n} X_t$ is *asymptotically efficient.* [Hint: the formula for $\hat{\mu}_{blue}$ is given in problem 7.2 of the book.]

7. **2-step-ahead prediction and iterated projection–case of AR(2)**. Let X_t satisfy a mean zero, causal AR(2) model: $X_t - \phi_1 X_{t-1} - \phi_2 X_{t-2} = Z_t$ where Z_t is the innovations white noise. The objective is to predict X_2 given data $X_s, s \leq 0$. Let $\hat{X}_1$ be the 1-step-ahead prediction, i.e., the projection of X_1 on $\bar{sp}\{X_s, s \leq 0\}$, and recall that $\hat{X}_1 = \phi_1 X_0 + \phi_2 X_{-1}$. Show that $P_{\bar{sp}(X_s, s \leq 0)} X_2 = \phi_1 \hat{X}_1 + \phi_2 X_0$, i.e., use the one-step ahead formula twice: once to predict X_1, and then plug in $\hat{X}_1$ instead of the unobserved X_1 in the one-step ahead formula to predict X_2. [Hint: verify the normal equations using the suggested formula.]

8. **2-step-ahead prediction and iterated projection–general case**. Let X_t be a mean zero, weakly stationary time series with spectral density $f(\lambda) > 0$ for all λ. The objective is to predict X_2 given data $X_s, s \leq 0$. Let $\hat{X}_1$ be the 1-step-ahead prediction, i.e., the projection of X_1 on $\bar{sp}\{X_s, s \leq 0\}$, and denote $\hat{X}_1 = \sum_{j=0}^{\infty} a_j X_{-j}$. Show that $P_{\bar{sp}(X_s, s \leq 0)} X_2 = a_0 \hat{X}_1 + a_1 X_0 + a_2 X_{-1} + \cdots$, i.e., you use the one-step ahead formula twice: once to predict X_1, and then plug in $\hat{X}_1$ instead of the unobserved X_1 in the one-step ahead formula to predict X_2. [Hint: use the AR(∞) representation.]

9. From Ch. 5 of Brockwell and Davis, do problem 1.

10. From Ch. 5 of Brockwell and Davis, do problem 20.

11. **Kolmogorov's formula**. The proof of Theorem 5.8.1 in the book proceeds by approximating an AR(∞) by a causal AR(p) with appropriately large order p, and then showing eq. (5.8.1) holds for all causal AR(p). Verify eq. (5.8.1) of Brockwell and Davis in the special case of a causal AR(1) model. [Hint: use the power series identity that is valid for all $x \in (-1, 1)$, i.e.,

$$\log(1 - 2x\cos(\lambda) + x^2) = -2 \sum_{k=1}^{\infty} \frac{\cos(\lambda k)}{k} x^k$$

where log is natural logarithm; hence, $\int_{-\pi}^{\pi} \log\left(1 - 2\phi\cos(\lambda) + \phi^2\right) d\lambda = 0$ if $\phi \in (-1, 1)$.]

12. **Spectral factorization**—this is a special case of the *Fejer-Riesz theorem.*
Let $\{X_t, t \in Z\}$ be a real-valued, mean zero, weakly stationary sequence with acf $\gamma(k)$ and spectral density $f(\lambda) = (2\pi)^{-1} \sum_k \gamma(k) e^{ik\lambda}$; for simplicity, assume $f(\lambda) > 0$ for all λ.

    If $\gamma(k) = 0$ for $|k| >$ some q, show that we can write $f(\lambda) = \frac{\tau^2}{2\pi} |\theta(e^{i\lambda})|^2$ where $\theta(z) = 1 + \theta_1 z + \cdots + \theta_q z^q$ for some $\tau^2 > 0$ and real coefficients θ_i. In other words, show that if $\gamma(k) = 0$ for $|k| > q$, then X_t has an MA(q) representation.

    [Hint: Define the agf $G(z) = \sum_{k=-q}^{q} \gamma(k) z^k$ and hence $f(\lambda) = (2\pi)^{-1} G(e^{i\lambda})$.]

    (a) Since $\gamma(k) = \gamma(-k)$, show that $G(z) = G(1/z)$. Use this to show that if z_j is a (complex) root of $z^q G(z) = 0$, then so is $1/z_j$.
    (b) Denote by $z_1, \ldots, z_k$ the roots of $z^q G(z) = 0$ that have modulus bigger or equal to one. Argue that $k = q$, and that there are no roots with modulus equal to one (since $f(\lambda) > 0$).
    (c) Express $G(z) = C z^{-q} \prod_{j=1}^{q} [(z - z_j)(z - 1/z_j)]$ for some constant C. Use this expression to identify τ^2 and a polynomial $\theta(z) = 1 + \theta_1 z + \cdots + \theta_q z^q$ having all roots outside the unit circle, such that $G(z) = \tau^2 \theta(z) \theta(1/z)$.
    (d) **MA(q) representation.** Show that $\{X_t, t \in Z\}$ has an MA(q) representation with respect to its own innovations. [Hint: construct a *whitening* filter with transfer function $1/\theta(z)$, and use it to *whiten* the series $\{X_t, t \in Z\}$.]

    NOTE: the Spectral factorization theorem stated above extends to the case $q = \infty$ but the general proof is different (as is the general way of calculating the MA coefficients).