On the Tail-sum Formula

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One learns in a first course in Probability Theory that if is X is a non-negative random variable then

$$\mathbf{E}[X] = \int_0^\infty \mathbf{P}[X > t]\, dt. \tag{1}$$

The discrete form of (1), namely

$$\mathbf{E}[X] = \sum_{n=1}^\infty \mathbf{P}[X \geq n], \tag{2}$$

for non-negative *integer-valued* X, is a special case of (1). More generally, if $\Psi \to [0,\infty[\to [0,\infty[$ is (locally) absolutely continuous with $\Psi(0) = 0$, with a.e. derivative $\psi \geq 0$, then

$$\mathbf{E}[\Psi(X)] = \int_0^\infty \psi(t)\mathbf{P}[X > t]\, dt. \tag{3}$$

Notice that there is no assumption that the integrals in question are finite; but, in all three cases, part of the assertion is that if one side of the equality is finite then so is the other.

Our purpose in this note is to point out that (1) and (3) remain valid when the probability $\mathbf{P}$ is replaced by an arbitrary measure. These formulas are typically proved by an application of Tonelli's form of the Fubini theorem on product measures, for which it is usually assumed that the factor measures are σ-finite (or at least countable sums of finite measures, as I learned long ago from R. Getoor). We shall see that in the present special case, the σ-finiteness hypothesis is unnecessary.

So let $(\Omega, \mathcal{F}, \mu)$ be a measure space and let $X : \Omega \to [0, \infty]$ be $\mathcal{F}$-measurable.

Proposition. *We have*

$$\int_\Omega X\, d\mu = \int_0^\infty \mu[X > t]\, dt. \tag{4}$$

Proof. Choose an increasing sequence of (finite) non-negative simple functions $\{X_n\}$ such that $X_n \uparrow X$ pointwise. Thus

$$X_n(\omega) = \sum_{k=1}^{k_n} c_{nk} \cdot 1_{B_{nk}}(\omega), \qquad \omega \in \Omega, n \geq 1.$$

with $0 < c_{nk} < \infty$ and $B_{nk} \in \mathcal{F}$. Without loss of generality we assume that the c_{nk} are distinct and the B_{nk} are pairwise disjoint, for each fixed n. Clearly

$$\int_\Omega X_n \, d\mu = \sum_{k=1}^{k_n} c_{nk} \mu(B_{nk}).$$

On the other hand $\mu[X_n > t] = \sum_{k: c_{nk} > t} \mu(B_{nk})$, and so

$$\begin{aligned}\int_0^\infty \mu[X_n > t] \, dt &= \int_0^\infty \sum_{k: c_{nk} > t} \mu(B_{nk}) \, dt \\ &= \sum_k \int_0^\infty 1_{\{c_{nk} > t\}} \, dt \cdot \mu(B_{nk}) \\ &= \sum_k c_{nk} \mu(B_{nk}).\end{aligned}$$

(The second equality in the above display follows from the familiar "semi-discrete" form of Tonelli, a consequence of the monotone convergence theorem.) Thus

$$\int_\Omega X_n \, d\mu = \int_0^\infty \mu[X_n > t] \, dt, \qquad n = 1, 2, \ldots.$$

To finish we let $n \to \infty$ and invoke the monotone convergence theorem on each side of the equality. ∎

The general version of (3), namely

$$\int_\Omega \Psi(X) \, d\mu = \int_0^\infty \psi(t) \mu[X > t] \, dt, \tag{5}$$

with Ψ and ψ as before, follows in similar fashion. [N.B. Because ψ is non-negative, Ψ is monotone increasing; we take $\Psi(X) :=\uparrow \lim_{t\to\infty} \Psi(t)$ on $\{X = \infty\}$.]