On a result of D.W. Stroock

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Recently, D.W. Stroock gave a simple probabilistic proof of L. Schwartz' "Borel graph theorem", which states (in the context of Banach spaces) that if E and F are separable Banach spaces and $\psi : E \to F$ is a linear map with Borel measurable graph, then ψ is continuous. In fact, Stroock obtained the continuity of ψ under the weaker hypothesis that ψ is μ-measurable for every centered Gaussian measure μ on E. My aim here is to show that Stroock's argument, slightly amended, proves an infinite dimensional version of the familiar fact [**4**] that Lebesgue measurable solutions of Cauchy's functional equation must be continuous (and linear).

A map $\psi : E \to F$ between Banach spaces in *additive* provided $\psi(x+y) = \psi(x) + \psi(y)$ for all $x, y \in E$. An additive ψ is necessarily linear over the rationals:

$$\psi(rx + sy) = r\psi(x) = s\psi(y), \qquad \forall r, s \in \mathbf{Q}, \forall x, y \in E, \tag{1}$$

(2) Theorem. *Let E and F be Banach spaces and let $\psi : E \to F$ be additive. If ψ is μ-measurable for every centered Gaussian (Radon) measure μ on E, then ψ is continuous (and linear).*

(We note that in Stroock's *proof*, the Gaussian measures used are all Radon measures, hence our slight relaxation of his measurability assumption.)

Let us begin with a brief discussion of Gaussian measures on Banach spaces. A probability measure μ on the Borel σ-algebra $\mathcal{B}(E)$ of a Banach space E is a *Radon* measure provided it is inner regular. Let $\mathcal{E} := \sigma\{x^* : x^* \in E^*\}$ denote the cylinder σ-algebra on E. A Radon probability measure μ on $\mathcal{B}(E)$ is uniquely determined by its restriction to $\mathcal{E}$, and $\mathcal{B}(E)$ is contained in the μ-completion $\mathcal{E}_\mu$ of $\mathcal{E}$; see[**1**; A.3.12].

A Radon probability measure μ on $\mathcal{B}(E)$ is a centered Gaussian measure if each $x^* \in E^*$, viewed as a random variable on the probability space $(E, \mathcal{E}_\mu, \mu)$, is normally distributed with mean 0 and variance $\sigma^2(x^*) \in [0, \infty)$. The following characterization of Gaussian Radon measures (due to X. Fernique) is crucial to Stroock's argument. Let μ be a Radon probability measure on $\mathcal{B}(E)$, and let X and Y be independent random elements of E with distribution μ (defined on some probability space $(\Omega, \mathcal{A}, \mathbf{P})$). If μ is centered Gaussian, then for each pair of real numbers (α, β) with $\alpha^2 + \beta^2 = 1$, the random vector $(\alpha X + \beta Y, \beta X - \alpha Y)$ has the same distribution as the pair (X, Y), namely the product measure $\mu \otimes \mu$ (on $\mathcal{E}_\mu \otimes \mathcal{E}_\mu$). (Notice that the map $(x, y) \mapsto (\alpha x + \beta y, \beta x - \alpha y)$ is $\mathcal{E}_\mu \otimes \mathcal{E}_\mu / \mathcal{E}_\mu \otimes \mathcal{E}_\mu$-measurable.) Conversely, if this equality in distribution holds for $\alpha = \beta = 1/\sqrt{2}$ alone, then μ is centered Gaussian.

Proof of Theorem 2. The proof in [**5**] needs to be supplemented at the two points where the full linearity of ψ is used: (i) in showing that the image $\psi_* \mu$ of a centered Gaussian measure μ on E is a centered Gaussian measure on F, and (ii) in the third display on page 6 of [**5**].

Let us take up point (ii) first. The display referred to makes use of the fact that

$$\langle \psi(tx), y^* \rangle = t \langle \psi(x), y^* \rangle, \qquad \forall t \in \mathbf{R}, x \in E, y^* \in F^*, \tag{3}$$

where F^* is the dual space of F. To see that this partial linearity follows from our hypotheses, fix $x \in E$ and consider the centered Gaussian Radon measure μ_x, the image of the standard normal distribution on $\mathbf{R}$ under the mapping $\mathbf{R} \ni t \mapsto tx \in E$. The assumed μ_x-measurability of ψ then implies that the additive function $f(t) := \langle \psi(tx), y^* \rangle$, $t \in \mathbf{R}$, is Lebesgue measurable. It is well known [**4**] that such an f is necessarily linear, and so (3) holds.

Turning to (i), we require the following simple fact.

(4) Lemma. *There is a sequence* $\{(\alpha_n, \beta_n) : n \geq 1\}$ *of pairs of rational numbers such that* $\alpha_n^2 + \beta_n^2 = 1$ *for all* n, *and* $\lim_n \alpha_n = \lim_n \beta_n = 1/\sqrt{2}$.

Proof. We produce the required pairs by an appeal to Euclid's construction of Pythagorean triples [**2**]. Abbreviate

$$\kappa := \sqrt{\frac{\sqrt{2}-1}{\sqrt{2}+1}},$$

choose a sequence of positive integers $m_n \in \{1, 2, \ldots, n-1\}$ such that

$$\lim_n \frac{m_n}{n} = \kappa,$$

and define rationals

$$\alpha_n := \frac{n^2 - m_n^2}{n^2 + m_n^2}, \qquad \beta_n := \frac{2m_n n}{n^2 + m_n^2}.$$

Clearly $\alpha_n^2 + \beta_n^2 = 1$ and $\lim_n \alpha_n = (1-\kappa^2)/(1+\kappa^2) = 1/\sqrt{2}$, as desired. □

We now fix a centered Gaussian Radon measure μ on $\mathcal{B}(E)$ and proceed to show that $\psi_*\mu$ is a centered Gaussian measure on F. Let X and Y be independent random elements of E with distribution μ. Let (α_n, β_n), $n \geq 1$, be as in Lemma 4. Then, using (1) for the first equality below,

$$\begin{aligned}(\alpha_n\psi(X) + \beta_n\psi(Y), \beta_n\psi(X) - \alpha_n\psi(Y)) &= (\psi(\alpha_n X + \beta_n Y), \psi(\beta_n X - \alpha_n Y)) \\ &\stackrel{d}{=} (\psi(X), \psi(Y)),\end{aligned}$$

the $\stackrel{d}{=}$ indicating equality in distribution. Sending $n \to \infty$ we obtain

$$\left(\frac{\psi(X)+\psi(Y)}{\sqrt{2}}, \frac{\psi(X)-\psi(Y)}{\sqrt{2}}\right) \stackrel{d}{=} (\psi(X), \psi(Y)),$$

so $\psi_*\mu$, the distribution of $\psi(X)$, is a centered Gaussian Radon probability measure on $\mathcal{B}(F)$. □

(5) Remark. Concerning the "universal Gaussian measurability" hypothesis in the Theorem, we follow [**5**] in noting that if $\psi : E \to F$ is additive and has a Borel measurable graph $G \subset E \times F$, and if E and F are separable, then $\psi^{-1}(B) = \pi_E(G \cap (E \times B))$ is an analytic subset of E (hence universally measurable) for each $B \in \mathcal{B}(F)$, so Theorem 2 applies. The separability condition on E is harmless, since the function ψ is continuous if and only if it is sequentially continuous.

References

[1] Bogachev, V.: *Gaussian Measures*, Mathematical Surveys and Monographs, **62**, American Mathematical Society, Providence, 1998.

[2] Euclid: *Elements*, Book X, Proposition 29, Lemma 1.

[3] Fernique, X.: Régularité des trajectoires des fonctions aléatoires gaussiennes, In *Écoles d' été de probabilités de Saint-Flour IV-1974*, Lecture Notes in Mathematics, **480**, Springer-Verlag, Berlin, 1975, pp. 1–96.

[4] Letac, G.: Cauchy functional equation again, *Amer. Math. Monthly* **85** (1978) 663–664.

[5] Stroock, D.W.: On a theorem of Laurent Schwartz, *C. R. Acad. Sci. Paris, Ser. I*, **349** (2011) 5–6.