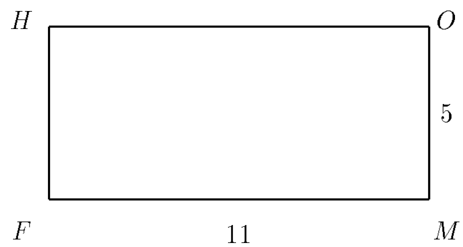
FIFTY-EIGHTH ANNUAL
WILLIAM LOWELL PUTNAM MATHEMATICAL COMPETITION

Saturday, December 6, 1997 — Examination A

A-1. A rectangle, $HOMF$, has sides $HO = 11$ and $OM = 5$. A triangle ABC has H as the intersection of the altitudes, O the center of the circumscribed circle, M the midpoint of BC, and F the foot of the altitude from A. What is the length of BC?



A-2. Players $1, 2, 3, \ldots, n$ are seated around a table, and each has a single penny. Player 1 passes a penny to player 2, who then passes two pennies to player 3. Player 3 then passes one penny to Player 4, who passes two pennies to Player 5, and so on, players alternately passing one penny or two to the next player who still has some pennies. A player who runs out of pennies drops out of the game and leaves the table. Find an infinite set of numbers n for which some player ends up with all n pennies.

A-3. Evaluate

$$\int_0^\infty \left(x - \frac{x^3}{2} + \frac{x^5}{2\cdot 4} - \frac{x^7}{2\cdot 4\cdot 6} + \cdots\right)\left(1 + \frac{x^2}{2^2} + \frac{x^4}{2^2\cdot 4^2} + \frac{x^6}{2^2\cdot 4^2\cdot 6^2} + \cdots\right).$$

A-4. Let G be a group with identity e and $\phi : G \to G$ a function such that

$$\phi(g_1)\phi(g_2)\phi(g_3) = \phi(h_1)\phi(h_2)\phi(h_3)$$

whenever $g_1g_2g_3 = e = h_1h_2h_3$. Prove that there exists an element $a \in G$ such that $\psi(x) = a\phi(x)$ is a homomorphism (i.e. $\psi(xy) = \psi(x)\psi(y)$ for all $x, y \in G$).

A-5. Let N_n denote the number of ordered n-tuples of positive integers $(a_1, a_2, \ldots, a_n)$ such that $1/a_1 + 1/a_2 + \ldots + 1/a_n = 1$. Determine whether N_{10} is even or odd.

A-6. For a positive integer n and any real number c, define x_k recursively by $x_0 = 0$, $x_1 = 1$, and for $k \geq 0$,

$$x_{k+2} = \frac{cx_{k+1} - (n-k)x_k}{k+1}.$$

Fix n and then take c to be the largest value for which $x_{n+1} = 0$. Find x_k in terms of n and k, $1 \leq k \leq n$.

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Saturday, December 6, 1997 Examination B

B-1. Let $\{x\}$ denote the distance between the real number x and the nearest integer. For each positive integer n, evaluate

$$F_n = \sum_{m=1}^{6n-1} \min(\{\frac{m}{6n}\}, \{\frac{m}{3n}\}).$$

(Here $\min(a, b)$ denotes the minimum of a and b.)

B-2. Let f be a twice-differentiable real-valued function satisfying

$$f(x) + f''(x) = -xg(x)f'(x),$$

where $g(x) \geq 0$ for all real x. Prove that $|f(x)|$ is bounded.

B-3. For each positive integer n, write the sum $\sum_{m=1}^{n} \frac{1}{m}$ in the form p_n/q_n, where p_n and q_n are relatively prime positive integers. Determine all n such that 5 does not divide q_n.

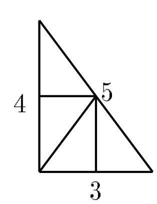
B-4. Let $a_{m,n}$ denote the coefficient of x^n in the expansion of $(1 + x + x^2)^m$. Prove that for all [integers] $k \geq 0$,

$$0 \leq \sum_{i=0}^{\lfloor \frac{2k}{3} \rfloor} (-1)^i a_{k-i,i} \leq 1.$$

B-5. Prove that for $n \geq 2$,

$$\overbrace{2^{2^{\cdot^{\cdot^{\cdot^{2}}}}}}^{n \text{ terms}} \equiv \overbrace{2^{2^{\cdot^{\cdot^{\cdot^{2}}}}}}^{n-1 \text{ terms}} \pmod{n}.$$

B-6. The dissection of the 3–4–5 triangle shown below has diameter 5/2.



Find the least diameter of a dissection of this triangle into four parts. (The diameter of a dissection is the least upper bound of the distances between pairs of points belonging to the same part.)