FORTY-EIGHTH ANNUAL
WILLIAM LOWELL PUTNAM MATHEMATICAL COMPETITION

Saturday, December 5, 1987 Examination A

A-1. Curves A, B, C, and D, are defined in the plane as follows:

$$A = \left\{(x,y) : x^2 - y^2 = \frac{x}{x^2+y^2}\right\},$$
$$B = \left\{(x,y) : 2xy + \frac{y}{x^2+y^2} = 3\right\},$$
$$C = \{(x,y) : x^3 - 3xy^2 + 3y = 1\},$$
$$D = \{(x,y) : 3x^2y - 3x - y^3 = 0\}.$$

Prove that $A \cap B = C \cap D$.

A-2. The sequence of digits

1 2 3 4 5 6 7 8 9 1 0 1 1 1 3 1 4 1 5 1 6 1 7 1 8 1 9 2 0 2 1 ...

is obtained by writing the positive integers in order. If the 10^nth digit in this sequence occurs in the part of the sequence in which the m-digit numbers are placed, define $f(n)$ to be m. For example $f(2) = 2$ because the 100th digit enters the sequence in the placement of the two-digit integer 55. Find, with proof, $f(1987)$.

A-3. For all real x, the real-valued function $y = f(x)$ satisfies

$$y'' - 2y' + y = 2e^x.$$

(a) If $f(x) > 0$ for all real x, must $f'(x) > 0$ for all real x? Explain.

(b) If $f'(x) > 0$ for all real x, must $f(x) > 0$ for all real x? Explain.

A-4. Let P be a polynomial, with real coefficients, in three variables and F be a function of two variables such that

$$P(ux, uy, uz) = u^2 F(y - x, z - x) \quad \text{for all real } x, y, z, u,$$

and such that $P(1,0,0) = 4$, $P(0,1,0) = 5$, and $P(0,0,1) = 6$. Also let A, B, C be complex numbers with $P(A,B,C) = 0$ and $|B - A| = 10$. Find $|C - A|$.

A-5. Let

$$\mathbf{G}(x,y) = \left(\frac{-y}{x^2+4y^2}, \frac{x}{x^2+4y^2}, 0\right).$$

Prove or disprove that there is a vector-valued function

$$\mathbf{F}(x,y,z) = (M(x,y,z), N(x,y,z), P(x,y,z))$$

with the following properties:

(i) M, N, P have continuous partial derivatives for all $(x,y,z) \neq (0,0,0)$;

(ii) $\operatorname{curl} \mathbf{F} = \mathbf{0}$ for all $(x,y,z) \neq (0,0,0)$;

(iii) $\mathbf{F}(x,y,0) = \mathbf{G}(x,y)$.

A-6. For each positive integer n, let $a(n)$ be the number of zeros in the base 3 representation of n. For which positive real numbers x does the series

$$\sum_{n=1}^{\infty} \frac{x^{a(n)}}{n^3}$$

converge?

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Saturday, December 5, 1987 Examination B

B-1. Evaluate:

$$\int_2^4 \frac{\sqrt{\ln(9-x)}\,dx}{\sqrt{\ln(9-x)}+\sqrt{\ln(x+3)}}.$$

B-2. Let r, s, and t be integers with $0 \le r$, $0 \le s$, and $r+s \le t$. Prove that

$$\frac{\binom{s}{0}}{\binom{t}{r}} + \frac{\binom{s}{1}}{\binom{t}{r+1}} + \frac{\binom{s}{2}}{\binom{t}{r+2}} + \cdots + \frac{\binom{s}{s}}{\binom{t}{r+s}} = \frac{t+1}{(t+1-s)\binom{t-s}{r}}.$$

B-3. Let F be a field in which $1+1 \neq 0$. Show that the set of solutions to the equation $x^2+y^2=1$ with x and y in F is given by $(x,y)=(1,0)$ and

$$(x,y) = \left(\frac{r^2-1}{r^2+1}, \frac{2r}{r^2+1}\right),$$

where r runs through the elements of F such that $r^2 \neq -1$.

B-4. Let $(x_1, y_1) = (0.8, 0.6)$ and let $x_{n+1} = x_n \cos y_n - y_n \sin y_n$ and $y_{n+1} = x_n \sin y_n + y_n \cos y_n$ for $n = 1, 2, 3, \ldots$. For each of $\lim_{n\to\infty} x_n$ and $\lim_{n\to\infty} y_n$, prove that the limit exists and find it or prove that the limit does not exist.

B-5. Let O_n be the n-dimensional zero vector (0,0,...,0). Let M be a $2n \times n$ matrix of complex numbers such that whenever $(z_1, z_2, \ldots, z_n)M = O_n$, with complex z_i, not all zero, then at least one of the z_i is not real. Prove that for arbitrary real numbers $r_1, r_2, \ldots r_{2n}$, there are complex numbers $w_1, w_2, \ldots, w_n$ such that

$$\text{Re}\left[M(w_1, \ldots, w_n)^t\right] = (r_1, \ldots, r_{2n})^t.$$

(Note: If C is a matrix of complex numbers, Re(C) is the matrix whose entries are the real parts of the entries of C.)

B-6. Let F be the field of p^2 elements where p is an odd prime. Suppose S is a set of $(p^2-1)/2$ distinct nonzero elements of F with the property that for each $a \neq 0$ in F, exactly one of a and $-a$ is in S. Let N be the number of elements in the intersection $S \cap \{2a : a \in S\}$. Prove that N is even.