FORTY-FIFTH ANNUAL
WILLIAM LOWELL PUTNAM MATHEMATICAL COMPETITION†

Saturday, December 1, 1984 — Examination A

A-1. Let A be a solid $a \times b \times c$ rectangular brick in three dimensions, where $a, b, c > 0$. Let B be the set of all points which are a distance at most one from some point of A (in particular, B contains A). Express the volume of B as a polynomial in a, b, and c.

A-2. Express $\displaystyle\sum_{k=1}^{\infty} \frac{6^k}{(3^{k+1} - 2^{k+1})(3^k - 2^k)}$ as a rational number

A-3. Let n be a positive integer. Let a, b, x be real numbers, with $a \neq b$, and let M_n denote the $2n \times 2n$ matrix whose (i,j) entry m_{ij} is given by

$$m_{ij} = \begin{cases} x & \text{if } i = j, \\ a & \text{if } i \neq j \text{ and } i+j \text{ is even}, \\ b & \text{if } i \neq j \text{ and } i+j \text{ is odd}. \end{cases}$$

Thus, for example, $M_2 = \begin{pmatrix} x & b & a & b \\ b & x & b & a \\ a & b & x & b \\ b & a & b & x \end{pmatrix}$. Express $\lim_{x \to a} \det M_n/(x-a)^{2n-2}$ as a polynomial in a, b, and n, where $\det M_n$ denotes the determinant of M_n.

A-4. A convex pentagon $P = ABCDE$, with vertices labeled consecutively, is inscribed in a circle of radius 1. Find the maximum area of P subject to the condition that the chords AC and BD be perpendicular.

A-5. Let R be the region consisting of all triples (x,y,z) of nonnegative real numbers satisfying $x + y + z \leq 1$. Let $w = 1 - x - y - z$. Express the value of the triple integral

$$\iiint_R x^1 y^9 z^8 w^4 \, dx \, dy \, dz$$

in the form $a!b!c!d!/n!$, where a, b, c, d, and n are positive integers.

A-6. Let n be a positive integer, and let $f(n)$ denote the last nonzero digit in the decimal expansion of $n!$. For instance, $f(5) = 2$.

(a) Show that is $a_1, a_2, \ldots, a_k$ are *distinct* nonnegative integers, then $f(5^{a_1} + 5^{a_2} + \cdots + 5^{a_k})$ depends only on the sum $a_1 + a_2 + \cdots + a_k$.

(b) Assuming part (a), we can define

$$g(s) = f(5^{a_1} + 5^{a_2} + \cdots + 5^{a_k}),$$

where $s = a_1 + a_2 + \cdots + a_k$. Find the least positive integer p for which

$$g(s) = g(s+p), \qquad \text{for all } s \geq 1,$$

or else show that no such p exists.

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Saturday, December 1, 1984 Examination B

B-1. Let n be a positive integer, and define

$$f(n) = 1! + 2! + \cdots n!.$$

Find polynomials $P(x)$ and $Q(x)$ such that

$$f(n+2) = P(n)f(n+1) + Q(n)f(n),$$

for all $n \geq 1$.

B-2. Find the minimum value of

$$(u-v)^2 + \left(\sqrt{2-u^2} - \frac{9}{v}\right)^2$$

for $0 < u < \sqrt{2}$ and $v > 0$.

B-3. Prove or disprove the following statement: If F is a finite set with two or more elements, then there exists a binary operation $*$ on F such that for all x, y, z in F,

(i) $x * z = y * z$ implies $x = y$ (right cancellation holds),

and

(ii) $x * (y * z) \neq (x * y) * z$ (*no* case of associativity holds).

B-4. Find, with proof, all real-valued functions $y = g(x)$ defined and *continuous* on $[0, \infty)$, positive on $(0, \infty)$, such that for all $x > 0$ the y-coordinate of the centroid of the region

$$R_x = \{(s,t) : 0 \leq s \leq x, 0 \leq t \leq g(s)\}$$

is the same as the average value of g on $[0, x]$.

B-5. For each nonnegative integer k, let $d(k)$ denote the number of 1's in the binary expansion of k (for example, $d(0) = 0$ and $d(5) = 2$). Let m be a positive integer. Express

$$\sum_{k=0}^{2^m-1} (-1)^{d(k)} k^m$$

in the form $(-1)^m a^{f(m)}(g(m))!$, where a is an integer and f and g are polynomials.

B-6. A sequence of convex polygons $\{P_n\}$, $n \geq 0$, is defined inductively as follows. P_0 is an equilateral triangle with sides of length 1. Once P_n has been determined, its sides are trisected; the vertices of P_{n+1} are the *interior* trisection points of the sides of P_n. Thus, P_{n+1} is obtained by cutting corners off P_n, and P_n has $3 \cdot 2^n$ sides. (P_1 is a regular hexagon with sides of length 1/3.)

Express $\lim_{n\to\infty} \text{Area}(P_n)$ in the form $\sqrt{a}/b$, where a and b are positive integers.