FORTY-THIRD ANNUAL
WILLIAM LOWELL PUTNAM MATHEMATICAL COMPETITION

Saturday, December 4, 1982 — Examination A

A-1. Let V be the region in the cartesian plane consisting of all points (x, y) satisfying the simultaneous equations

$$|x| \leq y \leq |x| + 3 \quad \text{and} \quad y \leq 4.$$

Find the centroid $(\overline{x}, \overline{y})$ of V.

A-2. For positive real x. let

$$B_n(x) = 1^x + 2^x + 3^x + \cdots + n^x.$$

Prove or disprove the convergence of

$$\sum_{n=2}^{\infty} \frac{B_n(\log_n 2)}{(n \log_2 n)^2}.$$

A-3. Evaluate

$$\int_0^{\infty} \frac{\operatorname{Arctan}(\pi x) - \operatorname{Arctan}(x)}{x}\, dx.$$

A-4. Assume that the system of simultaneous differential equations

$$y' = -z^3, \quad z' = y^3$$

with the initial conditions $y(0) = 1$, $z(0) = 0$ has a unique solution $y = f(x)$, $z = g(x)$ defined for all real x. Prove that there exists a positive constant L such that for all real x,

$$f(x + L) = f(x), \quad g(x + L) = g(x).$$

A-5. Let a, b, c, and d be positive integers and

$$r = 1 - \frac{a}{b} - \frac{c}{d}.$$

Given that $a + c \leq 1982$ and $r > 0$, prove that

$$r > \frac{1}{1983^3}.$$

A-6. Let σ be a bijection of the positive integers, that is, a one-to-one function from $\{1, 2, 3, \ldots\}$ onto itself. Let $x_1, x_2, x_3, \ldots$ be a sequence of real numbers with the following properties:

(i) $|x_n|$ is a strictly decreasing function of n;
(ii) $|\sigma(n) - n| \cdot |x_n| \to 0$ as $n \to \infty$;
(iii) $\lim_{n\to\infty} \sum_{k=1}^{n} x_k = 1$.

Prove or disprove that these conditions imply that

$$\lim_{n\to\infty} \sum_{k=1}^{n} x_{\sigma(k)} = 1.$$

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Saturday, December 4, 1982 Examination B

B-1. Let M be the midpoint of side BC of a general $\triangle ABC$. Using the *smallest possible* n, describe a method for cutting $\triangle AMB$ into n triangles which can be reassembled to form a triangle congruent to $\triangle AMC$.

B-2. Let $A(x, y)$ denote the number of points (m, n) in the plane with integer coordinates m and n satisfying $m^2 + n^2 \leq x^2 + y^2$. Let $g = \sum_{k=0}^{\infty} e^{-k^2}$. Express

$$\int_{-\infty}^{\infty} \int_{-\infty}^{\infty} A(x, y) e^{-x^2-y^2}\, dx\, dy$$

as a polynomial in g.

B-3. Let p_n be the probability that $c + d$ is a perfect square when the integers c and d are selected independently at random from the set $\{1, 2, 3, \ldots, n\}$. Show that $\lim_{n\to\infty}(p_n\sqrt{n})$ exists and express this limit in the form $r(\sqrt{s} - t)$, where s and t are integers and r is a rational number.

B-4. Let $n_1, n_2, \ldots n_s$ be distinct integers such that

$$(n_1 + k)(n_2 + k) \cdots (n_s + k)$$

is an integral multiple of $n_1 n_2 \cdots n_s$ for every integer k. For each of the following assertions, give a proof or a counterexample:

(a) $|n_i| = 1$ for some i.

(b) If further all n_i are positive, then

$$(n_1, n_2, \ldots, n_s) = (1, 2, \ldots, s).$$

B-5. For each $x > e^e$ define a sequence $S_x = u_0, u_1, u_2, \ldots$ recursively as follows: $u_0 = e$, while for $n \geq 0$, $n + n + 1$ is the logarithm of x to the base u_n. Prove that S_x converges to a number $g(x)$ and that the function g defined in this way is continuous for $x > e^e$.

B-6. Let $K(x, y, z)$ denote the area of a triangle whose sides have lengths x, y, and z. For any two triangles with sides a, b, c and a', b', c', respectively, prove that

$$\sqrt{K(a,b,c)} + \sqrt{K(a',b',c')} | le \sqrt{K(a+a', b+b', c+c')}$$

and determine the cases of equality.