FORTY-FOURTH ANNUAL
WILLIAM LOWELL PUTNAM MATHEMATICAL COMPETITION

Saturday, December 5, 1981 — Examination A

A-1. Let $E(n)$ denote the largest integer k such that 5^k is an integral divisor of the product $1^1 2^2 3^3 \cdots n^n$. Calculate

$$\lim_{n\to\infty} \frac{E(n)}{n}.$$

A-2. Two distinct squares of the 8 by 8 chessboard C are said to be adjacent if they have a vertex or side in common. Also, g is called a C-gap if for every numbering of C with all the integers $1, 2, \ldots, 64$ there exist two adjacent squares whose numbers differ by at least g. Determine the largest C-gap g.

A-3. Find

$$\lim_{t\to\infty} \left[e^{-t} \int_0^t \int_0^t \frac{e^x - e^y}{x - y}\, dx\, dy \right]$$

or show that the limit does not exist.

A-4. A point P moves inside a unit square in a straight line at unit speed. When it meets a corner it escapes. When it meets an edge its line of motion is reflected so that the angle of incidence equals the angle of reflection.

Let $N(T)$ be the number of starting directions from a fixed interior point P_0 for which P escapes within T units of time. Find the least constant a for which constants b and c exist such that

$$N(T) \le aT^2 + bT + c$$

for all $T > 0$ and all initial points P_0.

A-5. Let $P(x)$ be a polynomial with real coefficients and form the polynomial

$$Q(x) = (x^2+1)P(x)P'(x) + x\left([P(x)]^2 + [P'(x)]^2\right).$$

Given that the equation $P(x) = 0$ has n distinct real roots exceeding 1, prove or disprove that the equation $Q(x) = 0$ has at least $2n - 1$ distinct real roots.

A-6. Suppose that each of the vertices of $\triangle ABC$ is a lattice point in the (x, y)-plane and that there is exactly one lattice point P in the *interior* of the triangle. The line AP is extended to meet BC at E. Determine the largest possible value for the ratio of lengths of segments

$$\frac{|AP|}{|PE|}.$$

[A lattice point is a point whose coordinates x and y are integers.]

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Saturday, December 5, 1981 Examination B

B-1. Find

$$\lim_{n\to\infty}\left[\frac{1}{n^5}\sum_{h=1}^{n}\sum_{k=1}^{n}(5h^4-18h^2k^2+5k^4)\right].$$

B-2. Determine the minimum value of

$$(r-1)^2+\left(\frac{s}{r}-1\right)^2+\left(\frac{t}{s}-1\right)^2\left(\frac{4}{t}-1\right)^2$$

for all real numbers r, s, t with $1 \le r \le s \le t \le 4$.

B-3. Prove that there are infinitely many positive integers n with the property that if p is a prime divisor of n^2+3, then p is also a divisor of k^2+3 for some integer k with $k^2 < n$.

B-4. Let V be a set of 5 by 7 matrices, with real entries and with the property that $rA + sB \in V$ whenever $A, B \in V$ and r and s are scalars (i.e., real numbers). *Prove or disprove* the following assertion: If V contains matrices of ranks 0, 1, 2, 4, and 5, then it also contains a matrix of rank 3.

[The rank of a nonzero matrix M is the largest k such that the entries of some k rows and some k columns form a k by k matrix with nonzero determinant.]

B-5. Let $B(n)$ be the numbers of ones in the base two expression for the positive integer n. For example, $B(6) = B(110_2) = 2$ and $B(15) = B(1111_2) = 4$. Determine whether or not

$$\exp\left(\sum_{n=1}^{\infty}\frac{B(n)}{n(n+1)}\right)$$

is a rational number. Here $\exp(x)$ denotes e^x.

B-6. Let C be a fixed unit circle in the Cartesian plane. For any convex polygon P each of whose sides is tangent to C, let $N(P, h, k)$ be the number of points common to P and the unit cirle with center (h, k). Let $H(P)$ be the region of all points (x, y) for which $N(P, x, y) \ge 1$ and $F(P)$ be the area of $H(P)$. Find the smallest number u with

$$\frac{1}{F(P)}\iint N(P,x,y)\,dx\,dy < u$$

for all polygons P, where the double integral is taken over $H(P)$.