On a Characterization of the Poisson Process

P.J. Fitzsimmons

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We give a new proof of the following extension of a result of J. Wesołowski [**2**]. For the statement of the theorem we fix a filtered probability space $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\ge 0}, \mathbf{P})$ satisfying the usual conditions.

Theorem. *Let $X = (X_t)_{t\ge 0}$ be real-valued adapted process defined on $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\ge 0}, \mathbf{P})$, with cadlag increasing sample paths and $X_0 = 0$. Define $Y_t := X_t - t$. If each of the processes (i) $Y := (Y_t)_{t\ge 0}$, (ii) $L := (Y_t^2 - t)_{t\ge 0}$, and (iii) $M := (Y_t^3 - 3tY_t - t)_{t\ge 0}$ is a local martingale, then X is a unit-rate Poisson process.*

Proof. In view of a well-known theorem of S. Watanabe [**1**], we need only show that X is a pure jump process with all jumps equal to $+1$.

First observe that the local martingale Y has paths of locally finite variation. It is therefore the compensated sum of its jumps. These jumps are positive: $\Delta Y_t = Y_t - Y_{t-} = \Delta X_t$. It follows that X increases *only* by jumps: $X_t = \sum_{0<s\le t} \Delta X_s$. Next, by Itô's formula,

$$L_t + t = Y_t^2 = \int_0^t 2Y_{s-}\, dY_s + \sum_{0<s\le t} (\Delta Y_s)^2 = \int_0^t 2Y_{s-}\, dY_s + \sum_{0<s\le t} (\Delta X_s)^2,$$

so $K_t := \sum_{0<s\le t} (\Delta X_s)^2 - t$ is a local martingale. A second application of Itô's formula yields

$$M_t + t = Y_t^3 - 3tY_t = 3\int_0^t (Y_{s-}^2 - s)\, dY_s + 3\int_0^t Y_{s-}\, dK_s + \sum_{0<s\le t} (\Delta Y_s)^3.$$

It follows that $K'_t := \sum_{0<s\le t} (\Delta Y_s)^3 - t$ is a local martingale. Consequently,

$$\sum_{0<s\le t} \Delta X_s \cdot (\Delta X_s - 1)^2 = \sum_{0<s\le t} \big((\Delta X_s)^3 - 2(\Delta X_s)^2 + \Delta X_s\big) = K'_t - 2K_t + Y_t$$

is a local martingale *with increasing paths*; this local martingale therefore vanishes. Thus $\Delta X_s \in \{0, 1\}$ for all $s > 0$ almost surely, as desired. ∎

References

[1] S. Watanabe: On discontinuous additive functionals and Lévy measures of a Markov process. *Japan. J. Math.* **34** (1964) 53–70.

[2] J. Wesołowski: A martingale characterization of the Poisson process, *Bull. Polish Acad. Math.* **38** (1990) 49–53.