On the Kimura-Ruehr Identities

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The identities (1) and (2) below were posed by N. Kimura as a problem in the *American Mathematical Monthly* in 1979; the solution of O.G. Ruehr appears in [**KR80**], and is accomplished by a trigonometric substitution. (An alternative proof using Weierstrass' theorem and a comparison of moments is also suggested by Ruehr. This leads to certain identities involving sums of binomial coefficients, and a story of its own, for which see [**A18, A19, EZ19**].)

Here's the first identity:

$$\int_{-1/2}^{3/2} f(3x^2 - 2x^3)\,dx = 2\int_0^1 f(3x^2 - 2x^3)\,dx, \tag{1}$$

for all continuous $f : [0,1] \to \mathbf{R}$. A probabilist coming upon (1) naturally sees it as a statement that the image of the uniform distribution on $[-1/2, 3/2]$ under the mapping $x \mapsto 3x^2 - 2x^3$ coincides with the image of the uniform distribution on $[0,1]$ under (the restriction to $[0,1]$ of) this same mapping. One way to check this assertion is to show that the two images (both measures on the Borel subsets of $[0,1]$) have identical cumulative distribution functions. A sketch is all one needs to see that this equality of cdfs amounts to the statement that the sum of the three (distinct) roots of $2x^3 - 3x^2 + y = 0$ is $3/2$. But that sum is notoriously $-1/2$ times the coefficient of x^2 in the polynomial $2x^3 - 3x^2 + y$, namely $3/2$, as desired.

Kimura's second identity

$$\int_{-1/2}^{3/2} x f(3x^2 - 2x^3)\,dx = 2\int_0^1 x f(3x^2 - 2x^3)\,dx, \tag{2}$$

can be dealt with in much the same way by viewing the integrals on each side as "expectations" with respect to the image under $x \mapsto 3x^2 - 2x^3$ of the measure whose density with respect to Lebesgue measure is x (on the left side) and $2x$ (on the right). (It matters not that the measure on the left is a signed measure.) Consideration of cdfs now leads to the need to check that the sum of the *squares* of the roots of $2x^3 - 3x^2 + y = 0$ (for $0 < y < 1$) is equal to $9/4$. But the sum of the pairwise products of the roots vanishes because the coefficient of x is 0. Also, we noticed in examining the first identity that the sum of the roots is $3/2$, so the sum of squares must be $(3/2)^2 = 9/4$, as required.

There is nothing magical about the cubic $3x^2 - 2x^3$. Indeed, let p be a (real) cubic polynomial with 2 distinct critical points $a < b$. Suppose, without loss of generality, that $p(a) < p(b)$. Let A be the smallest x with $p(x) = p(b)$ and let B be the largest x with $p(x) = p(a)$. Then p maps both $[a, b]$ and $[A, B]$ onto $[p(a), p(b)]$ and

$$(b-a)\int_A^B f(p(x))\,dx = (B-A)\int_a^b f(p(x))\,dx,$$

and for an affine $g : [A, B] \to \mathbf{R}$,

$$(b-a)\int_A^B g(x)f(p(x))\,dx = (B-A)\int_a^b g(x)f(p(x))\,dx,$$

for all continuous $f : [p(a), p(b)] \to \mathbf{R}$. For proof repeat the above argument or scale the whole situation to fit.

References

[A18] Allouche, J.-P.: A generalization of an identity due to Kimura and Ruehr, *Integers* **18A** (2018) #A1, 6 pages.

[A19] Allouche, J.-P.: Two binomial identities of Ruehr revisited, *Amer. Math. Monthly* **126** (2019) 217–225.

[EZ19] Ekhad, S.B. and Zeilberger, Z.: Some remarks on a recent article by J.-P. Allouche,
`http://sites.math.rutgers.edu/~zeilberg/mamarim/mamarimPDF/allouche.pdf`

[KR80] Kimura, N. and Ruehr, O.G.: Change of variable formula for definite integrals, E 2765, *Amer. Math. Monthly* **87** (1980) 307–308.