Note on Kemeny's Constant

by

P. J. Fitzsimmons

In a recent Letter to the Editor [**1**] of the *Journal of Applied Probability*, O. Angel and M. Holmes have proved a conjecture found in [**2**] that Kemeny's constant is infinite in the case of a discrete time Markov chain with countably infinite state space. We give here a different proof, showing how to reduce the problem to the finite state space case, where Hunter [**3**] has provided a lower bound.

Let $X = (X_k)_{k \ge 0}$ be an positive recurrent Markov chain with countably infinite state space E. As such X admits a unique stationary distribution π. As usual, $T_y(X) := \inf\{k \ge 1 : X_k = y\}$ denotes the hitting time of state $y \in E$. It is well known that when E is finite the expected hitting time

$$\sum_{y \in E} \mathbf{E}^x[T_y(X)]\,\pi(y), \tag{1}$$

of a state chosen independently of X using π, is a constant not depending on the starting state x, the so-called Kemeny constant. For discussion and further references see [**2**] and [**3**]. Hunter in [**3**] has proved the lower bound

$$\sum_{y \in E} \mathbf{E}^x[T_y(X)]\,\pi(y) \ge (\mathrm{card}(E)+1)/2, \qquad x \in E, \tag{2}$$

but, as remarked in [**2**], was unable to extend the argument to the infinite state space case. The authors of [**1**] proved a local version of Hunter's lower bound, valid whether the state space is finite or infinite, and used that to prove the conjecture of [**2**] mentioned in the first paragraph.

Let $E_1 \subset E_2 \subset \cdots$ be an increasing sequence of finite subsets of E with union all of E, such that E_n has cardinality n. Fix n and consider the Markov chain $Y^{(n)}$ obtained by observing X only when it is in E_n. More precisely, for $X_0 = x \in E_n$ define $T_0^{(n)} := 0$, and $T_{k+1}^{(n)} := \min\{j > T_k^{(n)} : X_j \in E_n\}$, $k = 0, 1, 2, \ldots$. Then $Y_k^{(n)} := X_{T_k^{(n)}}$, $k = 0, 1, 2, \ldots$ defines a positive recurrent Markov chain on E_n.

By the ergodic theorem, if $A \subset E_n$, then

$$\lim_{m\to\infty} \frac{\sum_{k=1}^m 1_A(Y_k^{(n)})}{m} = \lim_{m\to\infty} \frac{\sum_{k=1}^{T_m^{(n)}} 1_A(X_k)}{\sum_{j=1}^{T_m^{(n)}} 1_{E_n}(X_j)} = \frac{\pi(A)}{\pi(E_n)},$$

$\mathbf{P}^x$-a.s. for $x \in E_n$. It follows that $[\pi(E_n)]^{-1}\pi$ is the stationary distribution for $Y^{(n)}$.

Evidently

$$T_y(Y^{(n)}) \le T_y(X)$$

for all $y \in E_n$. Consequently,

$$\sum_{y \in E_n} \mathbf{E}^x[T_x(Y^{(n)})]\,\pi(y) \le \sum_{y \in E_n} \mathbf{E}^x[T_y(X)]\,\pi(y) \le \sum_{y \in E} \mathbf{E}^x[T_y(X)]\,\pi(y), \qquad \forall x \in E_n,$$

and so by (2) applied to $Y^{(n)}$

$$\sum_{y\in E} \mathbf{E}^x[T_y(X)]\,\pi(y) \geq \pi(E_n)\cdot(n+1)/2, \qquad \forall x \in E_n.$$

Sending n off to infinity we find that

$$\sum_{y\in E} \mathbf{E}^x[T_y(X)]\,\pi(y) = \infty, \qquad \forall x \in E,$$

as desired.

References

[1] Angel, O., Holmes, M.: Kemeny's constant for infinite DTMCs is infinite, *J. Appl. Prob.* **56** (2019) 1269–1270.

[2] Bini, D., Hunter, J.J., Latouche, G., Meini, B, Taylor, P.G.: Why is Kemeny's constant a constant? *J. Appl. Prob.* **55** (2018) 1025–1036.

[H] Hunter, J.J.: Mixing times with applications to perturbed Markov chains, *Linear Algebra Appl.* **417** (2006) 108–123.