Remark on a note of Keller, Pinchover, and Pogorzelski on the discrete L^2 Hardy inequality

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In a recent note [**2**] in the *American Mathematical Monthly*, the authors demonstrate the following improvement of the classical L^2 Hardy inequality; to wit

$$\sum_{n=1}^{\infty} u(n)^2 \nu(n) \le \frac{1}{2} \sum_{n=1}^{\infty} [u(n) - u(n-1)]^2, \tag{1}$$

for functions $u : \{0, 1, 2, \ldots\} \to \mathbf{R}$ of finite support, with $u(0) = 0$. Here

$$\nu(n) := 1 - \frac{\sqrt{1 + n^{-1}} + \sqrt{1 - n^{-1}}}{2}, \qquad n = 1, 2, \ldots.$$

In the original Hardy inequality the weight on the left is $(1/8)n^{-2}$ whereas

$$\nu(n) = (1/8)n^{-2} + (5/128)n^{-4} + \cdots > (1/8)n^{-2}.$$

My goal here is to point out that (1) is a special case of a general inequality found in [**1**]. More precisely, apply [**1**; Thm. (1.9)(a)] when the basic process is the simple symmetric random walk on $\{0, 1, 2, \ldots\}$, in continuous time with unit rate holding times, with absorbing barrier at 0. The Dirichlet form of this process is

$$\mathcal{E}(u, u) = \frac{1}{2} \sum_{n=1}^{\infty} [u(n) - u(n-1)]^2,$$

the right side of (1). Meanwhile, because the infinitesimal generator of this process is the discrete Laplacian,

$$\mathcal{L}w(n) = \frac{1}{2}[w(n+1) - 2w(n) + w(n-1)],$$

the choice $w(n) := \sqrt{n}$ yields $\mathcal{L}w(n) + w(n)\nu(n) = 0$ for all $n \ge 1$. Thus Theorem (1.9)(a) of [**1**] can be applied (with $\delta = 1$ there) and (1) follows.

References

[1] P.J. Fitzsimmons: Hardy's inequality for Dirichlet forms. *J. Math. Anal. Appl.* **250** (2000) 548–560.

[2] M. Keller, Y. Pinchover, F. Pogorzelski: An improved discrete Hardy inequality. *MAA Monthly* **125** April 2018, 347–350.