Note on the Grüss Inequality

Let $(\Omega, \mathcal{F}, \mathbf{P})$ be a probability space, and let $X : \Omega \to \mathbf{R}$ be a bounded random variable with

$$m_X := \operatorname{ess\,inf} X, \qquad M_X := \operatorname{ess\,sup} X. \tag{1}$$

Proposition.

$$|\mathrm{Cov}(X,Y)| \le \frac{(M_X - m_X)(M_Y - m_Y)}{4}. \tag{2}$$

Proof. The general case follows immediately from the special case $X = Y$ and the Cauchy-Schwarz inequality. Thus we shall assume that $X = Y$ and show that

$$\mathrm{Var}(X) \le \frac{(M_X - m_X)^2}{4}. \tag{3}$$

In proving (3) we first assume that X has the following additional symmetry properties:

$$\mathbf{E}(X) = 0 \quad \text{and} \quad m_X = -M_X. \tag{4}$$

Then

$$\mathrm{Var}(X) = E(X^2) \le M_X^2 = [(M_X - m_X)/2]^2 = \frac{(M_X - m_X)^2}{4}. \tag{5}$$

In general, apply the preceding to $X^* := Z(X - c)$, where $c := (M_X + m_X)/2$ and Z is a random variable independent of X with $\mathbf{P}[Z = 1] = \mathbf{P}[Z = -1] = 1/2$. Clearly $M_{X^*} = -m_{X^*} = (M_X - m_X)/2$, and

$$\mathrm{Var}(X) \le \mathbf{E}\left[(X - c)^2\right] = \mathrm{Var}(X^*) \le \frac{(M_{X^*} - m_{X^*})^2}{4} = \frac{(M_X - m_X)^2}{4}. \tag{6}$$

$\square$

It is evident from (5) and (6) that equality holds in (3) if and only if

$$P[X^* = M_{X^*}] = P[X^* = m_{X^*}] = \frac{1}{2}, \tag{7}$$

or what amounts to the same thing

$$P[X \in \{M_X, m_X\}] = 1. \tag{8}$$

That is, equality holds in (3) if and only if the essential range of X contains at most two points.

References

[1] X. Li, R.N. Mohapatra, R.S. Rodriguez: Grüss-type inequalities, *J. Math. Anal. Appl.*, **267** (2002) 434–443.