On "Filtering of a reflected Brownian motion with respect to its local time" by G. Nappo & B. Torti

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1. Let $X = (X_t)_{t\geq 0}$ be a reflected Brownian motion, defined on a filtered probability space $(\Omega, \mathcal{F}, \mathcal{F}_t, \mathbf{P})$. Let $L = (L_t)_{t\geq 0}$ be the local time of X at 0, normalized so that $X - L$ is a standard Brownian motion. Define, for Borel $f : [0,\infty[\to [0,\infty[$,

$$K_u(f) := \int_0^\infty f(x\sqrt{u})x \exp(-x^2/2)\, dx, \qquad u \geq 0. \tag{1}$$

The following result is proved in [**1**], by an elementary (if tedious) discrete approximation argument. Formula (1) is in fact a special case of a general formula that follows readily from the excursion theory of Markov processes.

Theorem. *Let $\mathcal{G}_t := \sigma\{L_s : 0 \leq s \leq t\}$. Then*

$$\mathbf{P}[f(X_t)\,|\,\mathcal{G}_t] = K_{A(t)}(f), \qquad t \geq 0, \tag{2}$$

in which $A(t) = A_t := t - G_t$ and $G_t := \sup\{s \leq t : X_s = 0\}$, so that A_t is the age of the excursion of X from 0 that is in progress at time t.

2. Formula (2) is common to all strong Markov processes possessing a regular recurrent point—labelled 0 for convenience. For this paragraph let X denote a right Markov process with distinguished state 0, which we assume to be a regular recurrent point. We assume for simplicity that $\mathbf{P}^0[X_t = 0] = 0$ for each $t > 0$. There is then a CAF $L = (L_t)_{t\geq 0}$, the local time of X at 0, unique up to a constant multiple, and the Itô excursion measure $\mathbf{n}$ such that

$$\mathbf{P}^0 \sum_{s\in G} Z_s F_s(e_s) = \mathbf{P}^0 \int_0^\infty Z_s \mathbf{n}(F_s)\, dL_s, \tag{3}$$

for every optional $Z \geq 0$ and every positive measurable function F defined on the product of $[0,\infty[$ and the space of excursion paths. In this formula, G is the set of left endpoints of excursion intervals, and e_s is the excursion beginning at time $s \in G$: $e_s(t) = X_{t+s}$, $0 \leq t \leq \zeta_s$, where $\zeta_s = \zeta(e_s)$ is the lifetime of e_s.

In the present general context we define

$$K_u(f) := \mathbf{n}[f(e(u))\,|\,\zeta > u], \qquad u > 0. \tag{4}$$

Proof of (2). For $t > 0$, The left endpoint G_t is characterized as the unique element $s \in G$ such that $s \le t$ and $\zeta(e_s) > t - s$. Thus, if $Z \ge 0$ is optional,

$$\begin{aligned}
\mathbf{P}^0[Z_{G(t)} f(X_t)] &= \mathbf{P}^0 \sum_{s \in G, s \le t} Z_s f(e_s(t-s)) 1_{\{\zeta(e_s) > t-s\}} \\
&= \mathbf{P}^0 \int_0^t Z_s \mathbf{n}[f(e_s(t-s)); \zeta(e_s) > t-s] \, dL_s \\
&= \mathbf{P}^0 \int_0^t Z_s K_{t-s}(f) \mathbf{n}[\zeta(e_s) > t-s] \, dL_s \\
&= \mathbf{P}^0 \sum_{s \in G, s \le t} Z_s K_{t-s}(f) 1_{\{\zeta(e_s) > t-s\}} \\
&= \mathbf{P}^0[Z_{G(t)} K_{t-G(t)}(f)].
\end{aligned}$$

We conclude that $\mathbf{P}^0[f(X_t) \,|\, \mathcal{F}_{G(t)}] = K_{t-G(t)}(f)$. This proves formula (2) (in the general context of the present paragraph) because $\mathcal{G}_t \subset \mathcal{F}_{G(t)}$ and G_t is $\mathcal{G}_t$-measurable (being the last point of increase of L prior to time t). □

3. The explicit expression (1) for K_u in the special case of reflected Brownian motion follows from the well-known formula

$$\mathbf{n}[e(u) \in dx; \zeta > u] = \frac{2x}{\sqrt{2\pi u^3}} \exp(-x^2/2u), \qquad x > 0. \tag{5}$$

References

[1] G. Nappo, B. Torti: Filtering of a reflected Brownian motion with respect to its local time, *Stoch. Proc. Appl.* **116** (2006) 568-584.