Drift Transformations of Symmetric Diffusions, and Duality

by

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ABSTRACT

Starting with a symmetric Markov diffusion process X (with symmetry measure m and $L^2(m)$ infinitesimal generator A) and a suitable core $\mathcal{C}$ for the Dirichlet form of X, we describe a class of derivations defined on $\mathcal{C}$. Associated with each such derivation B is a drift transformation of X, obtained through Girsanov's theorem. The transformed process X^B is typically non-symmetric, but we are able to show that if the "divergence" of B is positive, then m is an excessive measure for X^B, and the $L^2(m)$ infinitesimal generator of X^B is an extension of $f \mapsto Af + B(f)$. The methods used are mainly probabilistic, and involve the notions of even and odd continuous additive functionals, and Nakao's stochastic divergence.

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1. Introduction

Consider the drift perturbation of the probabilist's Laplacian $\frac{1}{2}\Delta$ by a locally square-integrable vector field $B : \mathbf{R}^d \to \mathbf{R}^d$; *i.e.*, the operator

$$\mathcal{A}^B f := \frac{1}{2}\Delta f + B \cdot \nabla f \tag{1.1}$$

acting on the class $C_c^\infty(\mathbf{R}^d)$ of smooth compactly-supported real-valued functions on $\mathbf{R}^d$. As is well known, a diffusion process associated with $\mathcal{A}^B$ can be constructed by means of a Girsanov transformation of Brownian motion. Indeed, let $X = (\Omega, \mathcal{F}, \mathcal{F}_t, \theta_t, X_t, \mathbf{P}^x)$ be the canonical realization of Brownian motion in $\mathbf{R}^d$, and define a local martingale continuous additive functional of X by the Itô integral

$$M_t^B := \int_0^t B(X_s) \cdot dX_s. \tag{1.2}$$

Now use the stochastic exponential

$$L_t^B := \exp(M_t^B - \tfrac{1}{2}\langle M^B \rangle_t)$$

as a Radon-Nikodym factor to define (locally) a measure P_B^x,

$$P_B^x|_{\mathcal{F}_t \cap \{t<\zeta\}} := L_t^B \cdot \mathbf{P}^x|_{\mathcal{F}_t}, \tag{1.3}$$

for each $x \in \mathbf{R}^d$. The need for the lifetime ζ on the left side of (1.3) stems from the fact that the (positive) local martingale L^B need not be a martingale. Let X^B denote the diffusion with laws P_B^x, $x \in \mathbf{R}^d$. As a consequence of Girsanov's theorem, one can show that if $f \in C_c^\infty(\mathbf{R}^d)$, then

$$f(X_t^B) - f(X_0^B) - \int_0^t \mathcal{A}^B(X_s^B)\, ds, \qquad t \geq 0, \tag{1.4}$$

is a P_B^x-martingale for each $x \in E$. Moreover, if $\operatorname{div} B \geq 0$, then the Lebesgue measure m on $\mathbf{R}^d$ is an excessive measure for the transition semigroup $(P_t^B)_{t\geq 0}$ of X^B; that is $mP_t^B \leq m$ for all $t > 0$ and $\lim_{t\downarrow 0} mP_t^B(C) = m(C)$ for all Borel sets C. Consequently, (P_t^B) acts acts a strongly continuous contraction semigroup in $L^2(m)$, and the associated $L^2(m)$-infinitesimal generator A^B is an extension of $\mathcal{A}^B$, as is easily deduced from the fact that the process displayed in (1.4) is a martingale.

Now suppose that $\operatorname{div} B = 0$ in the distribution sense. Let X^{-B} denote the diffusion obtained from the above construction with $-B$ substituted for B. In this case X^B and X^{-B} are in duality with respect to Lebesgue measure:

$$(P_t^{-B} f, g)_m = (f, P_t^B g)_m, \qquad \forall f, g \in L^2(m), \tag{1.5}$$

where $(P_t^{-B})_{t\geq 0}$ is the transition semigroup of X^{-B} and $(u,v)_m := \int_{\mathbf{R}^d} uv\, dm$ is the natural inner product in $L^2(m)$. Of course, the infinitesimal generator of (P_t^{-B}) is an extension of the operator

$$\mathcal{A}^{-B} f = \tfrac{1}{2}\Delta f - B \cdot \nabla f, \qquad f \in C_c^\infty(\mathbf{R}^d). \tag{1.6}$$

In [**St96**], Stannat has discussed drift transformations of the above type in an infinite dimensional setting. A problem left open (at the end of section 1(d) in [**St96**]) is the question of whether the adjoint relationship (1.5) persists in that setting. One of our goals in this paper is to show that this is the case. In fact we show that the story told in the first two paragraphs of this introduction holds up for any symmetric diffusion admitting a "nice" *carré du champs* operator. We devote the balance of this section to an informal discussion of the basic ideas behind our results.

Let $X = (X_t, \mathbf{P}^x)$ now denote a Markov diffusion process with values in some reasonable metric space E. For simplicity, we assume in the following discussion that the lifetime of X is infinite. By the term "Markov diffusion" we mean a path-continuous strong Markov process with quasi-left-continuous natural filtration $(\mathcal{F}_t)$. In particular, every local martingale over X has continuous paths. We assume in addition that the (predictable) quadratic variation $\langle M\rangle$ of any X-local martingale M is absolutely continuous with respect to Lebesgue measure. Thus, if M is a local martingale *and* a continuous additive functional (CAF) of X, then $\langle M\rangle_t = \int_0^t \Gamma(M)(X_s)\,ds$ for some function $\Gamma(M)\colon E \to [0,\infty[$. This representation can be polarized to yield a bilinear mapping $(M, M') \mapsto \Gamma(M, M')$ such that $\langle M, M'\rangle_t = \int_0^t \Gamma(M, M')(X_s)\,ds$ for any pair (M, M') of local martingale CAFs.

Let $Y = (Y_t, Q^x)$ be a second E-valued Markov diffusion with infinite lifetime, and suppose that Y is locally absolutely continuous with respect to X in the sense that

$$Q^x|_{\mathcal{F}_t} \ll \mathbf{P}^x|_{\mathcal{F}_t}, \qquad \forall\, t \geq 0.$$

The Radon-Nikodym density process $L_t := d(Q^x|_{\mathcal{F}_t})/d(\mathbf{P}^x|_{\mathcal{F}_t})$ is then a multiplicative functional (MF) of X and a strictly positive martingale. As such, L admits an exponential representation

$$L_t = \exp(M_t - \tfrac{1}{2}\langle M\rangle_t), \qquad t \geq 0, \tag{1.7}$$

where M is a local martingale CAF of X. Let A^X denote the infinitesimal generator of X, so that for suitable $f\colon E \to \mathbf{R}$,

$$M_t^f := f(X_t) - f(X_0) - \int_0^t A^X f(X_s)\,ds, \qquad t \geq 0,$$

is a local martingale CAF of X. Itô's formula informs us that $B\colon f \mapsto \Gamma(M, M^f)$ is a derivation, and (at least formally), the infinitesimal generator of Y is given by the formula

$$A^Y f = A^X f + B(f). \tag{1.8}$$

Conversely, given a suitable derivation B, there is an associated local martingale CAF of X, call it M^B, such that $\Gamma(M^B, M^f) = B(f)$ for all "smooth" f. Of course, if we substitute this local martingale for M on the right side of (1.7), then the generator of the corresponding process is given by (1.8). All of this is well known from the work of Kunita [**K69**].

Define, for smooth f, the derivation $B_f\colon g \mapsto \Gamma(M^f, M^g)$. In our general context B_f plays the role that ∇f plays in the context of Brownian motion; indeed, the mapping $f \mapsto B_f$ has the properties one would expect of a gradient operator. When X is symmetric, Nakao [**Na85**] has provided us with a way to compute the associated divergence operator.

Indeed let us now assume that X is symmetric with respect to some σ-finite measure m on E. That is, the transition semigroup (P_t) of X is self-adjoint on $L^2(m)$. Under this symmetry condition, given a local martingale CAF M of X, there is a CAF $\Lambda(M)$ that is locally of zero quadratic variation such that

$$(1.9)\qquad [f,\Lambda(M)] := \lim_{t\downarrow 0} t^{-1}\mathbf{P}^m[f(X_0)\Lambda(M)_t] = -\tfrac{1}{2}\textstyle\int_E \Gamma(M^f, M)\,dm, \qquad \forall f \in \mathcal{C},$$

where $\mathcal{C}$ is a suitable "core" for the Dirichlet space of X. Notice that if $\Lambda(M)$ is absolutely continuous (*i.e.*, $\Lambda(M)_t = \int_0^t \lambda(M)(X_s)\,ds$) then $[f,\Lambda(M)] = (f,\lambda(M))_m$, provided $\lambda(M) \in L^1_{\text{loc}}$. Specializing (1.9) to local martingales of the form M^B (B a derivation) we arrive at Nakao's "partial integration" formula

$$(1.10)\qquad \int_E B(f)\,dm = -[f, \operatorname{div} B], \qquad \forall f \in \mathcal{C},$$

where $\operatorname{div} B := 2\Lambda(M^B)$. The reader is invited to check that if X is Brownian motion and B is the derivation $f \mapsto b\cdot\nabla f$ induced by a vector field $b\colon \mathbf{R}^d \to \mathbf{R}^d$, then M^B_t is the Itô integral $\int_0^t b(X_s)\cdot dX_s$ and $(\operatorname{div} B)_t = \int_0^t [\operatorname{div} b](X_s)\,ds$ provided the distribution-sense divergence of b is a locally integrable function.

The dénouement of our story is provided by the observation made in [**Fi95**] that if M is any local martingale CAF of the symmetric process X, then $M_t + \Lambda(M)_t$ is anti-symmetric with respect to the natural time reversal involution $r_t\colon \omega \mapsto \omega(t-s)_{0\le s\le t}$ on the path space of X. Moreover, $\Lambda(M)_t$ is symmetric with respect to r_t. These facts allows us to "compute" the adjoint of the semigroup of the process X^B obtained from the Girsanov transformation of X by means of the martingale MF $L^B_t := \exp(M^B_t - \frac{1}{2}\langle M^B\rangle_t)$.

Indeed, suppose that B is a derivation such that $\operatorname{div} B \ge 0$, in the sense that $\Lambda(M^B)$ is an increasing CAF. In this case $\operatorname{div} B$ may be identified with a certain smooth measure on E (the Revuz measure of $\Lambda(M^B)$). Moreover, the positivity of $\operatorname{div} B$ implies that (P^B_t) is a strongly continuous contraction semigroup in $L^2(m)$. The adjoint semigroup $(\hat{P}^B_t)$ is associated with a certain right Markov process $\hat{X}^B$ (the dual process of X^B with respect to m), and working at the path-space level one sees that $\hat{X}^B$ is the Girsanov transformation of X by means of the martingale MF $\hat{L}^B_t := L^B_t \circ r_t$. The preceding discussion should make it clear that $\hat{L}^B_t$ is none other than $L^{-B}_t \exp(-(\operatorname{div} B)_t)$. Consequently, the infinitesimal generator of $(\hat{P}^B_t)$ is an extension of

$$\hat{\mathcal{A}}^B f := Af - B(f) - f \operatorname{div} B.$$

This yields the Brownian motion result discussed earlier when $\operatorname{div} B = 0$.

Let us now briefly outline the organization of the paper. In section 2 we describe in detail the setting of the paper and we record some preliminary results. Section 3 contains the proof of the main results outlined above. In a final section we record some examples illustrating the results presented in section 3.

For background on symmetric Markov process and Dirichlet spaces the reader can consult [**BH91, FOT94, MR92**]. Let us mention here a few specifics concerning notation: If $(F, \mathcal{F})$

is a measurable space, then $p\mathcal{F}$ and $b\mathcal{F}$ denote the classes of positive and bounded real-valued $\mathcal{F}$-measurable functions from F to $\mathbf{R}$; these prefixes have the same meaning when attached to other function classes. If μ is a measure on $(F, \mathcal{F})$ and $f : F \to [0, \infty]$ is $\mathcal{F}$-measurable, then $\mu(f)$ denotes the integral $\int_F f \, d\mu$ while $f \cdot \mu$ denotes the measure whose density with respect to μ is f. The term "additive functional" should be interpreted in the sense of [**FOT94**; p. 181]; that is, as "additive functional with an exceptional set of starting points." If E is a domain in Euclidean space, then $C_b^\infty(E)$ denotes the class of smooth real-valued functions $f : E \to \mathbf{R}$ such that f and its partial derivatives of all orders are bounded. The subclass $C_c^\infty(E)$ consists of those elements of $C_b^\infty(E)$ with compact (in E) support.

2. Preliminaries

We now describe precisely the setting in which we shall be working. The hypotheses set down below will be in force throughout the paper.

Let $(E, \mathcal{B}(E))$ be a Lusin metrizable topological space, *i.e.,* E is homeomorphic to a Borel subset of some compact metric space and $\mathcal{B}(E)$ is the class of Borel sets in E. Let m be a σ-finite measure on E and let $(\mathcal{E}, \mathcal{D})$ be a quasi-regular [**MR92**; IV.3] Dirichlet form on $L^2(m)$. We assume that $(\mathcal{E}, \mathcal{D})$ is *strongly local* in the sense that

$$(2.1) \qquad F, G \in C_c^\infty(\mathbf{R}), F \text{ constant on the support of } G \quad \Longrightarrow \quad \mathcal{E}(F_0 \circ u, G_0 \circ u) = 0, \ \forall u \in \mathcal{D},$$

where $F_0 := F - F(0), G_0 = G - G(0)$. Under these conditions there is a right Markov process $X = (\Omega, \mathcal{F}, \mathcal{F}_t, X_t, \theta_t, \mathbf{P}^x)$ that is symmetric with respect to m and properly associated with $(\mathcal{E}, \mathcal{D})$. Consequently, if (P_t) denotes the transition semigroup of X, then

$$(2.2) \qquad (f, P_t g)_m = (P_t f, g)_m \qquad \forall f, g \in L^2(m),$$

where $(u, v)_m := \int_E uv \, dm$. Viewed as a semigroup operating in $L^2(m)$, (P_t) is strongly continuous, and $P_t f$ is $\mathcal{B}(E)$-measurable for each bounded $\mathcal{B}(E)$-measurable function $f : E \to \mathbf{R}$. Quasi-regularity and (2.1) implies that X is a *diffusion* in the following strong sense:

(i) The $\mathbf{P}^m$-completion $(\mathcal{F}_t)_{t \geq 0}$ of the natural filtration $\sigma\{X_s; 0 \leq s \leq t\}$, $t \geq 0$, is quasi-left-continuous and the lifetime of X, denoted ζ, is an $(\mathcal{F}_t)$ *predictable* stopping time;

(ii) $t \mapsto X_t$ is continuous on $[0, \zeta[$ $\mathbf{P}^m$-a.s.

Consequently, every $(\mathcal{F}_t)$-stopping time is predictable, and every $(\mathcal{F}_t)$-martingale has continuous paths ($\mathbf{P}^m$-a.s.). See [**Sh88**; §47] and [**MR92**; IV.3].

We can (and do) take the sample space Ω to be the space of paths ω from $[0, \infty[$ to $E \cup \{\Delta\}$ that are E-valued and continuous on $[0, \zeta(\omega)[$ and that hold the value $\Delta \notin E$ after time $\zeta(\omega)$. As usual, any function f defined on E is automatically extended to the cemetery state Δ by the convention $f(\Delta) = 0$.

The Dirichlet form $(\mathcal{E}, \mathcal{D})$ is related to the transition semigroup (P_t) by

$$\mathcal{D} = \{u \in L^2(m) : \sup_{t > 0} \frac{1}{t}(u, u - P_t u)_m < \infty\};$$

$$\mathcal{E}(u,v) = \lim_{t\to 0} \frac{1}{t}(u, v - P_t v)_m, \qquad u,v \in \mathcal{D}.$$

Endowed with the inner product $\mathcal{E}_1(u,v) := \mathcal{E}(u,v) + (u,v)_m$, $\mathcal{D}$ is a Hilbert space.

The infinitesimal generator $(A, D(A))$ associated with $(\mathcal{E},\mathcal{D})$ is the operator in $L^2(m)$ determined by

$$Af := \lim_{t\to 0}[P_t f - f]/t, \qquad f \in D(A), \tag{2.3}$$

where the limit in question is taken in the (strong) $L^2(m)$ sense, and the domain $D(A)$ is precisely the vector space of functions for which the indicated limit exists. Notice that

$$(f, -Ag)_m = \mathcal{E}(f,g), \qquad f \in \mathcal{D}, g \in D(A). \tag{2.4}$$

Finally, we assume that X admits a *carré du champs* operator. Let $\mathcal{L}$ denote the vector space of real-valued $\mathcal{B}_E$-measurable functions on E.

(2.5) Hypothesis. There is an algebra $\mathcal{C} \subset D(A) \cap C_b(E)$ and a symmetric bilinear form $\Gamma: \mathcal{C}\times\mathcal{C} \to \mathcal{L}$ such that

(i) $\mathcal{C}$ is $\mathcal{E}_1$-dense in $\mathcal{D}$;

(ii) If $\varphi \in C_b^\infty(\mathbf{R})$ satisfies $\varphi(0) = 0$, then $\varphi\circ u \in \mathcal{C}$ for all $u \in \mathcal{C}$;

(iii) For all $u \in \mathcal{C}$ there exists $g \in \mathcal{C}$ such that $gu = u$;

(iv) $\Gamma(u,u) \geq 0$ for all $u \in \mathcal{C}$;

(v) $\Gamma(\varphi\circ u, \psi\circ v) = \varphi'\circ u \cdot \psi'\circ v \cdot \Gamma(u,v)$ for all $u, v \in \mathcal{C}$ and all $\varphi, \psi \in C_b^\infty$ such that $\varphi(0) = \psi(0) = 0$;

(vi) $\Gamma(u,v) = A(uv) - uAv - vAu$, m-a.e., for all $u, v \in \mathcal{C}$.

Because of (2.5)(i)(vi), Γ extends to a bilinear form $\Gamma : \mathcal{D}\times\mathcal{D} \to L^1(m)$, and $(f,g) \mapsto \Gamma(f,g)$ is a continuous mapping of $\mathcal{D}\times\mathcal{D}$ into $L^1(m)$. Moreover, Γ provides a disintegration of the form $\mathcal{E}$:

$$\mathcal{E}(u,v) = \frac{1}{2}\int_E \Gamma(u,v)\,dm, \qquad u,v \in \mathcal{D}. \tag{2.6}$$

See [**BH91**; I.4]. In the sequel, we shall often write $\Gamma(f)$ instead of $\Gamma(f,f)$. Notice that because of (2.5)(iv),

$$|\Gamma(f,g)|^2 \leq \Gamma(f)\cdot\Gamma(g) \tag{2.7}$$

identically if $f, g \in \mathcal{C}$, and a.e if $f, g \in \mathcal{D}$.

(2.8) Remarks. (a) One often starts with a bilinear form Γ satisfying (2.5)(iv)(v), defined on a "core" of functions $\mathcal{C}$ as in (2.5). If the "pre-Dirichlet form" given on $\mathcal{C}\times\mathcal{C}$ by the right side of (2.6) is well-defined and closable, then one can form its closure $(\mathcal{E},\mathcal{D})$. If this Dirichlet form is quasi-regular, then one is in the situation outlined above.

(b) Condition (2.5)(iii) may be thought of as the assumption that each element of $\mathcal{C}$ has "stochastically compact" support. Indeed, it can be shown that if $u \in \mathcal{C}$ then $\sup\{t : u(X_t) \neq 0\} <$

ζ on $\{\zeta < \infty\}$, $\mathbf{P}^m$-a.s. However, it should be noted that in typical infinite-dimensional contexts one has $m(E) < \infty$ and $1 \in \mathcal{C}$, in which case (2.5)(iii) is vacuous.

Let us recall briefly the probabilistic significance of the existence of the carré du champs operator Γ. Each element $u \in \mathcal{D}$ admits an m-modification $\tilde{u}$ (a quasi-continuous version) such that $t \mapsto \tilde{u}(X_t)$ is continuous on $[0, \infty[$, $\mathbf{P}^m$-a.s. We then have Fukushima's decomposition [**FOT94**; Thm. 5.2.2]:

$$\tilde{u}(X_t) - \tilde{u}(X_0) = M_t^u + N_t^u, \quad \forall t \geq 0, \mathbf{P}^x\text{-a.s. for q.e. } x \in E, \tag{2.9}$$

where M^u and N^u are continuous additive functionals (CAFs) of X, M^u is a martingale such that $\sup_{t>0} t^{-1}\mathbf{P}^m[(M_t^u)^2] < \infty$, and $\lim_{t\to 0} t^{-1}\mathbf{P}^m[(N_t^u)^2] = 0$. This decomposition is unique. In the sequel we shall refer to M^u as the martingale part of u. Given $u \in \mathcal{D}$, the martingale CAF M^u admits a quadratic variation process $\langle M^u \rangle$; *i.e.*, $\langle M^u \rangle$ is predictable and $(M^u)^2 - \langle M^u \rangle$ is a martingale CAF. Because of the quasi-left-continuity of the filtration of X, $\langle M^u \rangle$ is a PCAF. The Revuz measure of $\langle M^u \rangle$ (the so-called energy measure of u) is the smooth measure $\Gamma(u) \cdot m$; in other words, $\langle M^u \rangle_t = \int_0^t \Gamma(u)(X_s)\, ds$. Given two elements u and v of $\mathcal{D}$, the quadratic covariation $\langle M^u, M^v \rangle$ is a (signed) CAF of X with Revuz measure $\Gamma(u, v) \cdot m$, as can be seen by a simple polarization argument. Condition (2.5)(v) is a natural reflection of Itô's formula.

We shall use $\mathcal{M}$ to denote the class of martingale CAFs of finite energy. More precisely, define the mutual energy $\mathbf{e}(M, M')$ of two martingale CAFs M and M' by

$$\mathbf{e}(M, M') := \lim_{t\downarrow 0}(2t)^{-1}\mathbf{P}^m[M_t M_t'] = \lim_{t\downarrow 0}(2t)^{-1}\mathbf{P}^m\langle M, M' \rangle_t, \tag{2.10}$$

whenever the limit exists, and write $\mathbf{e}(M) = e(M, M)$. Then

$$\begin{aligned} \mathcal{M} := \{M : &M \text{ is a CAF of } X,\ \mathbf{P}^x[M_t^2] < \infty \text{ and} \\ &\mathbf{P}^x[M_t] = 0 \text{ for quasi-every } x \in E,\ \mathbf{e}(M) < \infty\}. \end{aligned} \tag{2.11}$$

When equipped with the inner product $(M, M') \mapsto \mathbf{e}(M, M')$, $\mathcal{M}$ becomes a Hilbert space; the associated norm is $\mathbf{e}(\cdot)^{1/2}$.

It is well known that the existence of a carré du champs operator implies that for each $M \in \mathcal{M}$ the quadratic variation process $\langle M \rangle$ is absolutely continuous with respect to Lebesgue measure; see, for example, [**DM87**; XV.26]. Consequently, there is a bilinear mapping $(M, N) \to \Gamma(M, N)$ from $\mathcal{M}$ into $L^1(m)$ such that

$$\langle M, N \rangle_t = \int_0^t \Gamma(M, N)(X_s)\, ds, \qquad t \geq 0, \forall\, M, N \in \mathcal{M}. \tag{2.12}$$

In particular,

$$e(M, N) = \tfrac{1}{2} \int_E \Gamma(M, N)\, dm, \qquad \forall M, N, \in \mathcal{M}. \tag{2.13}$$

Of course, $\Gamma(M^u, M^v) = \Gamma(u, v)$ if $u, v \in \mathcal{D}$, so our somewhat abusive notation should not cause confusion.

We write $\mathcal{M}_\mathcal{C}$ for the class of martingale CAFs of the form

$$\sum_{i=1}^{n} f_i * M^{g_i}, \qquad n \in \mathbf{N}, f_i \in \mathcal{C} \cup \{1\}, g_i \in \mathcal{C}, \tag{2.14}$$

where $f * M^g$ denotes the Itô integral $t \mapsto \int_0^t f(X_s)\, dM_s^g$. Clearly $\mathcal{M}_\mathcal{C} \subset \mathcal{M}$, and it can be shown that $\mathcal{M}_\mathcal{C}$ is **e**-dense in $\mathcal{M}$; *cf.* [**FOT94**; Lem. 5.6.3]. Notice that if $M = \sum_i f_i * M^{g_i} \in \mathcal{M}_\mathcal{C}$, then

$$\Gamma(M) = \sum_{i,j} f_i f_j \Gamma(g_i, g_j),$$

where $\Gamma(M) := \Gamma(M, M)$.

3. Drifts

Recall that $\mathcal{L}$ denotes the class of real-valued $\mathcal{B}_E$-measurable functions defined on E.

(3.1) Definition. A mapping $B: \mathcal{C} \to \mathcal{L}$ is said to be Γ-*bounded* provided there is a function $\beta \in p\mathcal{L}$ such that

$$|B(f)(x)| \le \beta(x) \cdot [\Gamma(f)(x)]^{1/2}, \qquad \forall x \in E, \forall f \in \mathcal{C}. \tag{3.2}$$

A function β as in (3.2) will be referred to as a Γ-*bound* for B.

Notice that condition (2.5)(v) implies that $\mathcal{C} \ni f \mapsto \Gamma(f, h)$ is a *derivation* for each fixed $h \in \mathcal{C}$: $\Gamma(fg, h) \equiv f\Gamma(g, h) + g\Gamma(f, h)$ for all $f, g \in \mathcal{C}$.

(3.3) Proposition. *If $B: \mathcal{C} \to \mathcal{L}$ is Γ-bounded and linear, then B is a derivation.*

Proof. It suffices to show that $B(f^2) \equiv 2fB(f)$ for all $f \in \mathcal{C}$. Fix $f \in \mathcal{C}$. Then

$$|B(f^2)(x)| \le \beta(x)[\Gamma(f^2)(x)]^{1/2} = \beta(x) \cdot 2|f(x)|[\Gamma(f)(x)]^{1/2}, \quad \forall x \in E,$$

so $B(f^2)(x) = 2f(x)B(f)(x)$ is certainly true whenever $f(x) = 0$. To handle the non-zero values of f, we use (2.5)(iii) to choose $g \in \mathcal{C}$ such that $gf = f$.

(3.4) Lemma. $\Gamma(g) \equiv 0$ *on* $\{f \ne 0\}$.

Proof of the Lemma. For any positive integer n we have, by (2.5)(v),

$$\Gamma(f) = \Gamma(g^n f) = g^{2n}\Gamma(f) + 2nfg^{2n-1}\Gamma(g, f) + n^2 f^2 g^{2n-2}\Gamma(g).$$

On $\{f \ne 0\} \subset \{g = 1\}$ this becomes

$$0 = 2n\Gamma(g, f) + n^2 f\Gamma(g).$$

Dividing through by n^2 and then sending $n \to \infty$, we see that $f\Gamma(g) = 0$ on $\{f \ne 0\}$. □

Continuing with the proof of the Proposition, define, for non-zero $c \in \mathbf{R}$,

$$f_c(x) := f(x) - cg(x), \qquad x \in E.$$

Clearly $f_c \in \mathcal{C}$. Let us compute $B(f_c^2)$ on $\{f \neq 0\}$. On the one hand, since $gf = f$,

$$B(f_c^2) = B(f^2) - 2cB(f) + c^2B(g^2) = B(f^2) - 2cB(f), \tag{3.5}$$

because $|B(g^2)| \le \beta\Gamma(g^2)^{1/2} = \beta \cdot 2|g|\Gamma(g) = 0$ on $\{f \neq 0\}$ in view of (3.4). On the other hand,

$$|B(f_c^2)| \le \beta\Gamma(f_c^2)^{1/2} = 2\beta|f_c|\Gamma(f_c)^{1/2}, \tag{3.6}$$

which vanishes on $\{f = c\} \subset \{f \neq 0\}$. Thus, when $f(x) = c \neq 0$, (3.5) and (3.6) yield

$$0 = B(f^2)(x) - 2cB(f)(x) = B(f^2)(x) - 2f(x)B(f)(x),$$

as desired. ▯

(3.7) Definition. We shall say that a Γ-bounded linear map $B: \mathcal{C} \to \mathcal{L}$ is a *smooth derivation* provided there is a choice of the Γ-bound β such that the measure $\beta^2 \cdot m$ is smooth in the sense of [**FOT94**; p. 80]. This amounts to assuming that the integral $\int_0^t \beta(X_s)^2\,ds$ defines a PCAF of X.

If $g \in \mathcal{C}$, then $f \mapsto \Gamma(g,f)$ is a Γ-bounded derivation with bound $\Gamma(g)^{1/2} \in L^2(m)$; each such mapping is therefore a smooth derivation in the sense of (3.7).

We write $\mathcal{M}_{\rm loc}$ for the class of local martingale CAFs of X. Since X is a diffusion, a localization argument shows that Γ extends to a bilinear mapping of $\mathcal{M}_{\rm loc} \times \mathcal{M}_{\rm loc}$ into $\mathcal{L}^1_{\rm loc}$ such that $\langle M, N\rangle_t = \int_0^t \Gamma(M,N)(X_s)\,ds$ for all $M, N \in \mathcal{M}_{\rm loc}$. Here and below, $\mathcal{L}^p_{\rm loc}$ denotes the vector space of (equivalence classes of) m-measurable functions q such that $|q|^p \cdot m$ is a smooth measure.

If $M \in \mathcal{M}_{\rm loc}$, then $f \mapsto \Gamma(M, M^f)$ is a linear mapping of $\mathcal{C}$ into $\mathcal{L}^2_{\rm loc}$, and it is Γ-bounded (m-a.e.) with bound $\Gamma(M)^{1/2} \in \mathcal{L}^2_{\rm loc}$. Thus, ignoring the fact that $\Gamma(M, M^f)$ is an equivalence class of functions, $f \mapsto \Gamma(M, M^f)$ is a smooth derivation. Conversely, we have the following

(3.8) Proposition. *Let B be a smooth derivation. Then there is a unique local martingale CAF M^B such that*

$$\Gamma(M^B, M^f) = B(f), \qquad m\text{-a.e. }, \forall f \in \mathcal{C}. \tag{3.9}$$

Moreover,

$$\Gamma(M^B) \le \beta^2 \qquad m\text{-a.e.} \tag{3.10}$$

If $g \in b\mathcal{B}_E$ then $gB: f \mapsto g \cdot B(f)$ is a smooth derivation and $M^{gB} = g * M^B$. If, in addition, $g\beta \in L^2(m)$, then $M^{gB} \in \mathcal{M}$.

Proof. Let $M = \sum_i f_i * M^{g_i}$ be a typical element of $\mathcal{M}_{\mathcal{C}}$. It is easy to check that if $\sum_{i,j} f_i(x)f_j(x)\Gamma(g_i,g_j)(x) = 0$ then $\sum_i f_i(x)B(g_i)(x) = 0$. This observation allows us to define a linear mapping $\Psi : \mathcal{M}_{\mathcal{C}} \to \mathcal{L}$ by setting

$$\Psi\left(\sum_i f_i * M^{g_i}\right)(x) := \sum_i f_i(x)B(g_i)(x), \qquad x \in E.$$

Notice that if $M = \sum_i f_i * M^{g_i} \in \mathcal{M}_\mathcal{C}$ and $h \in \mathcal{C}$ then $\Psi(h * M) = h\Psi(M)$. Also,

$$(3.11) \qquad |\Psi(M)| \le \beta \cdot \Gamma(M)^{1/2}, \qquad \forall M \in \mathcal{M}_\mathcal{C}.$$

Because $\beta^2 \cdot m$ is a smooth measure by hypothesis, it follows from (a slight extension of) a theorem of Kunita [**K69**; Prop. 2.4] that there is a unique $M^B \in \mathcal{M}_{\text{loc}}$ such that

$$(3.12) \qquad \Gamma(M^B, M) = \Psi(M) \qquad m\text{-a.e., } \forall M \in \mathcal{M}_\mathcal{C}.$$

Taking $M = M^f$ $(f \in \mathcal{C})$ in (3.12) we obtain (3.9).

Since $M^B \in \mathcal{M}_{\text{loc}}$, its quadratic variation process $\langle M^B \rangle_t = \int_0^t \Gamma(M^B)(X_s)\, ds$ is a PCAF of X with (smooth) Revuz measure $\Gamma(M^B) \cdot m$. Let (F_n) be a generalized nest (in the sense of [**FOT94**; p. 81]) such that $\int_{F_n} \Gamma(M^B)\, dm < \infty$ for all n. Then $1_{F_n} * M^B \in \mathcal{M}$ by a simple application of Doob's L^2 maximal inequality. Now $M \mapsto \Gamma(M)$ is a continuous mapping of $\mathcal{M}$ into $L^1(m)$, so (3.11) permits us to uniquely extend Ψ to all of $\mathcal{M}$ by continuity; this being done, (3.11) and (3.12) hold for all $M \in \mathcal{M}$. In particular,

$$1_{F_n}\Gamma(M^B) = \Gamma(M^B, 1_{F_n} * M^B) = \Psi(1_{F_n} * M^B) \le \beta\Gamma(1_{F_n} * M^B)^{1/2} = 1_{F_n}\beta\Gamma(M^B)^{1/2},$$

m-a.e., from which it follows that $\Gamma(M^B) \le \beta^2$, m-a.e. on $\cup_n F_n$. This yields (3.10) since (F_n) is a generalized nest.

If $g \in b\mathcal{B}_E$ then gB is a smooth derivation and the equality $M^{gB} = g * M^B$ follows easily from (3.12) and the density of $\mathcal{M}_\mathcal{C}$. The final assertion is now an immediate consequence of Doob's inequality $\square$

Let B be a smooth derivation with associated local martingale CAF M^B. Let $E \setminus E_B$ be a (Borel) properly exceptional set [**FOT94**; p. 134] that is a common exceptional set for the CAFs M^B and $\langle M^B \rangle$. Define the stochastic exponential

$$(3.13) \qquad L_t^B := \exp(M_t^B - \tfrac{1}{2}\langle M^B \rangle_t).$$

This is a positive supermartingale multiplicative functional of X and a local martingale on $[0, \zeta[$ under any of the laws $\mathbf{P}^x$, $x \in E_B$. By standard theory (e.g. [**Sh88**; § 62]) there is a family $(P_B^x)_{x \in E_B}$ of probability laws on $(\Omega, \mathcal{F}, \mathcal{F}_t)$ such that

$$(3.14) \qquad \mathbf{P}_B^x[F; t < \zeta] = \mathbf{P}^x[F \cdot L_t^B; t < \zeta], \qquad F \in b\mathcal{F}_t, t \ge 0, x \in E_B.$$

For emphasis, we use $(X_t^B)_{t \ge 0}$ to denote the coordinate process under the P_B^x. We complete the definition of X^B by taking $\mathbf{P}_B^x$ to be the point mass concentrated on the constant path $t \mapsto x$ in case $x \in E \setminus E_B$. We refer to X^B as the Girsanov transformation of X induced by the smooth derivation B. The reader can easily check that X^B is a diffusion in the sense outlined below (2.2), except that (P_t^B) need not preserve Borel measurability. This minor annoyance which can be remedied by redefining X^B on a suitably chosen Borel properly exceptional Borel set containing $E \setminus E_B$; for the details of such a modification see [**Fi01**; Cor. 3.23].

As a consequence of Girsanov's theorem, if $f \in \mathcal{C}$ and if $g := Af + B(f)$, then

$$\tilde{M}_t^f := f(X_t^B) - f(X_0^B) - \int_0^t g(X_s^B)\, ds$$

is a $\mathbf{P}_B^x$-local martingale on $[0, \zeta[$ for q.e. $x \in E$. Moreover, the quadratic variation of $\tilde{M}^f$ is the same as that of M^f; namely

$$\langle \tilde{M}^f \rangle_t = \int_0^t \Gamma(f)(X_s^B)\, ds, \qquad \forall t \geq 0, \mathbf{P}_B^x\text{-a.s.}$$

for q.e. $x \in E$.

(3.15) Definition. An *admissible drift* is a smooth derivation B such that

(a) $\beta \in L^2(|g| \cdot m)$ for all $g \in \mathcal{C}$, and

(b) $\int_E B(f)\, dm \leq 0$ for all $f \in \mathcal{C}$.

(3.16) Remarks. (a) Suppose $f \in \mathcal{C}$ and choose $g \in \mathcal{C}$ such that $gf = f$. Then, by (3.4), $|B(f)| = |B(gf)| \leq |g|\beta\Gamma(f)^{1/2}$ is dominated by a constant multiple of $|g|\beta$. Consequently, condition (3.15)(a) implies that $B(f) \in L^1(m)$ whenever $f \in \mathcal{C}$, so that (3.15)(b) is meaningful.

(b) In concrete situations, condition (a) above amounts to the hypothesis that β is locally square integrable. As we shall see in a moment, condition (b) is precisely the condition that m be a supermedian measure for the process X^B. It is the simplest condition ensuring that the transition semigroup (P_t^B) of X^B acts as a contraction semigroup in $L^2(m)$.

The following result extends [**St96**; Thm. 1.1] to our (broader) context; the approach taken in this paper are quite different from those used in [**St96**]. Both theorems concern the construction of a diffusion process whose L^2-generator is an extension of the operator $(\mathcal{A}^B, \mathcal{C})$ defined below. In [**St96**], a strongly continuous transition semigroup whose generator extends $(\mathcal{A}^B, \mathcal{C})$ is obtained and then the associated diffusion process is constructed. The diffusion is then shown to be a Girsanov transformation of X. We take the opposite tack, using the Girsanov transformation to construct the process X^B and then identifying its generator as an extension of $(\mathcal{A}^B, \mathcal{C})$. No uniqueness of the extension is claimed in either case; nonetheless, the semigroups and processes constructed in the two papers are the same. This follows from the discussion on pages 237–242 of [**St96**]. A uniqueness result is proved in [**St96**; Thm. 1.2], under the condition that $\mathcal{C}$ is dense in $D(A)$ with respect to the graph norm. We shall not take up this point here, but merely note that the proof in [**St96**] is robust enough to cover the context of the present paper.

(3.17) Theorem. *Let B be an admissible drift. Then*

(a) $mP_t^B \leq m$ for all $t > 0$ and $\lim_{t\to 0} mP_t^B = m$, setwise. That is, m is an excessive measure for X^B.

(b) $(P_t^B)_{t\geq 0}$ is a strongly continuous contraction semigroup in $L^2(m)$.

(c) Define a linear operator $\mathcal{A}^B : \mathcal{C} \to L^2(m)$ by

$$\mathcal{A}^B f := \mathcal{A}f + B(f), \qquad f \in \mathcal{C}.$$

Then $(A^B, D(A^B))$*, the* $L^2(m)$*-infinitesimal generator of* (P_t^B)*, is an extension of* $(\mathcal{A}^B, \mathcal{C})$*.*

The proof of Theorem (3.17) is based on a time-reversal argument developed in [**Fi95**]; we quickly recall the main ideas.

Given a path ω with $\zeta(\omega) > t$, define the reversed (at time t) path $r_t\omega$ by

$$r_t(\omega)(u) = \begin{cases} \omega(t-u), & 0 \le u \le t, \\ \omega(0), & u > t. \end{cases}$$

Then the symmetry (2.1) is reflected in the identity

$$\mathbf{P}^m[F\circ r_t; t < \zeta] = \mathbf{P}^m[F; t < \zeta], \tag{3.18}$$

which is valid for every positive $\mathcal{F}_t$-measurable function F on Ω and each $t > 0$.

(3.19) Definition. Given a CAF D of X, the *even* and *odd* parts of D are defined by the formulae

$$D_t^{\text{even}} = [D_t + D_t\circ r_t]/2 \qquad D_t^{\text{odd}} = [D_t - D_t\circ r_t]/2, \qquad t < \zeta.$$

We say that D is even (resp. odd) provided $D_t = D_t^{\text{even}}$ (resp. $D_t = D_t^{\text{odd}}$) $\mathbf{P}^m$-a.s. on $\{t < \zeta\}$ for each $t > 0$.

The following is drawn from [**Fi95**; Thm. 2.1].

(3.20) Proposition. *Every CAF of* X *that is locally of zero energy is even. In particular every CAF of locally finite variation is even.*

Let $\mathcal{N}$ (resp. $\mathcal{N}_{\text{loc}}$) denote the class of CAFs of X of zero energy (resp. locally of zero energy). From the work of Nakao, we know that there is a linear mapping $\Lambda\colon \mathcal{M} \to \mathcal{N}$ uniquely determined by the formula

$$\lim_{t\downarrow 0} t^{-1}\mathbf{P}^m[u(X_0)\Lambda(M)_t] = -\mathbf{e}(M^f, M), \qquad \forall u \in b\mathcal{D}, M \in \mathcal{M}.$$

The local nature of Λ [**Na85**; Lem. 3.5] allows us to extend Λ uniquely to a mapping of $\mathcal{M}_{\text{loc}}$ into $\mathcal{N}_{\text{loc}}$; we use the same symbol Λ to denote this extension.

In view of (3.20), $\Lambda(M)$ is an even CAF for each $M \in \mathcal{M}_{\text{loc}}$. For us the importance of Λ stems from the observation made in [**Fi95**; Cor. 3.1] that $M + \Lambda(M)$ is odd for each $M \in \mathcal{M}_{\text{loc}}$. We now use Λ to to compute the "dual" of the multiplicative function L^B. Since $\Lambda(M^B)$ is even and $M^B + \Lambda(M^B)$ is odd,

$$\begin{aligned} M_t^B\circ r_t &= [M_t^B + \Lambda(M^B)_t]\circ r_t - \Lambda(M^B)_t \\ &= -[M_t^B + \Lambda(M^B)_t] - \Lambda(M^B)_t = -M_t^B - 2\Lambda(M^B)_t. \end{aligned}$$

Thus, since $\langle M^B \rangle$ is even,

$$L_t^B\circ r_t = \exp(-M_t^B - \langle M^B\rangle_t)\exp(-2\Lambda(M)_t) = L_t^{-B}\exp(-2\Lambda(M)_t), \tag{3.21}$$

$\mathbf{P}^m$-a.s. on $\{t < \zeta\}$ for each $t > 0$. The extreme right term in (3.21) is a supermartingale MF of X, which we denote $\hat{L}^B$.

(3.22) Lemma. *If B is an admissible drift, then $\Lambda(M^B)$ is a PCAF of X.*

Proof. Pick a sequence $(f_n) \subset p\mathcal{C}$ such that $\{f_n > 0\}$, $n \in \mathbf{N}$ is a nest. (For example, fix a strictly positive 1-excessive function $u \in \mathcal{D}$, choose a sequence $(f_n^\circ) \subset p\mathcal{C}$ such that $\lim_n f_n^\circ = u$ q.e., and put $f_n := \sum_{i=1}^n f_n^\circ$.) Pick $g_n \in \mathcal{C}$ such that $g_n f_n = f_n$. Then, by the final assertion in (3.8), $g_n * M^B \in \mathcal{M}$ for each n. If $f \in \mathcal{C}$ then

$$0 \geq \int_E B(g_n f)\, dm = 2\mathbf{e}(g_n * M^B, M^f) = -2 \lim_{t \downarrow 0} t^{-1} \mathbf{P}^m [f(X_0) \Lambda(g_n * M^B)_t]. \tag{3.23}$$

Referring to the construction [**Na85**; (3.5)] of Λ we see that $\Lambda(g_n * M^B)_t = N_t^{\gamma_n} - \int_0^t \gamma_n(X_s)\, ds$, where $\gamma_n \in \mathcal{D}$ is uniquely determined by the fact that $\mathcal{E}_1(\gamma_n, f) = \mathbf{e}(g_n * M^B, M^f)$ for all $f \in b\mathcal{D}$. It follows from this and (3.23) that

$$0 \leq \lim_{t \downarrow 0} t^{-1} \mathbf{P}^m [f(X_0) \Lambda(g_n * M^B)_t] = \mathcal{E}_1(f, \gamma_n), \qquad \forall f \in \mathcal{C}. \tag{3.24}$$

In view of (3.24) and the density of $\mathcal{C}$ in $\mathcal{D}$, we can (and do) take γ_n to be 1-excessive, in which case $\gamma_n = U^1(\nu_n)$ for some (positive) measure ν_n of finite energy integral. But then $N_t^{\gamma_n} - \int_0^t \gamma_n(X_s)\, ds$ is just the PCAF with Revuz measure ν_n, by [**FOT94**; Lem. 5.4.1]. It follows that $\Lambda(g_n * M^B)$ is a PCAF for each $n \in \mathbf{N}$. Let τ_n denote the first exit time from $\{f_n > 0\}$. As noted already, Λ is "local" [**Na85**; Lem. 3.5]; since $g_n \equiv 1$ on $\{f_n > 0\}$ we therefore have $\Lambda(M^B)_t = \Lambda(g_n * M^B)_t$ for $0 \leq t < \tau_n$ ($\mathbf{P}^m$-a.s.). Since $\{f_n > 0\}$, $n \in \mathbf{N}$, is a nest, $\Lambda(M^B)$ is increasing on $[0, \zeta[$, $\mathbf{P}^m$-a.s. Consequently, $\Lambda(M^B)$ is a PCAF. ◻

Proof of Theorem (3.17). (a) If $f \in p\mathcal{B}_E$, then

$$\begin{aligned} mP_t^B(f) &= \mathbf{P}^m[L_t^B f(X_t)] = \mathbf{P}^m[L_t^B \circ r_t f(X_0)] \\ &= \mathbf{P}^m[f(X_0) \exp(-M_t^B - \tfrac{1}{2}\langle M^B \rangle_t) \exp(-2\Lambda(M)_t)] \\ &\leq \mathbf{P}^m[f(X_0) \exp(-M_t^B - \tfrac{1}{2}\langle M^B \rangle_t)] \\ &\leq \mathbf{P}^m[f(X_0)] = m(f). \end{aligned}$$

(In the above calculation, the second equality results from (3.14) and the third from (3.21); the first inequality holds since $\Lambda(M^B)_t$ is positive, and the second holds since L^{-B} is a positive supermartingale.) Thus, $mP_t^B \leq m$. The setwise convergence of mP_t^B to m follows from the fact that X^B is a right Markov process; see [**DM87**; XII.37b], for example. This proves part (a) of Theorem (3.17).

(b) Since m is excessive for (P_t^B), we have

$$\|P_t^B f\|_2^2 = \int_E [P_t^B f]^2\, dm \leq \int P_t^B(f^2)\, dm = mP_t^B(f^2) \leq m(f^2),$$

so (P_t^B) is a semigroup of $L^2(m)$-contractions. To see that (P_t^B) is strongly continuous, it suffices to show that the image of $L^2(m)$ under the 1-resolvent operator $U_B^1 := \int_0^\infty e^{-t} P_t^B\, dt$ is dense in $L^2(m)$. But if $f \in L^2(m)$ and $(f, U^1 g)_m = 0$ for all $g \in L^2(m)$, then the measures $f^+ \cdot m$ and $f^- \cdot m$ admit the same σ-finite 1-potential: $(f^+ \cdot m)U^1 = (f^- \cdot m)U^1$. By the uniqueness of charges property enjoyed by any right Markov process [**G90**; (2.12)], we must have $f^+ \cdot m = f^- \cdot m$, whence $f = 0$, m-a.e.

(c) Fix $f \in \mathcal{C}$ and choose $g \in p\mathcal{C}$ such that $gf = f$. Then, by Lemma (3.4), $B(f) = B(gf) = gB(f)$, so $\int_E [B(f)]^2 \, dm \le \|g\Gamma(f)\|_\infty \int_E \beta^2 g \, dm < \infty$. Thus $\mathcal{A}^B f \in L^2$. As noted above,

$$\tilde{M}_t^f := f(X_t^B) - f(X_0^B) - \int_0^t \mathcal{A}^B f(X_s^B) \, ds, \qquad 0 \le t < \zeta, \tag{3.25}$$

is a local martingale (on $[0, \zeta[$) CAF of X^B with quadratic variation $\langle \tilde{M}^f \rangle_t = \int_0^t \Gamma(f)(X_s^B) \, ds$. Since

$$Q^m(\langle \tilde{M}^f \rangle_t) = Q^m \int_0^t \Gamma(f)(X_s^B) \, ds = \int_0^t m P_s^B(\Gamma(f)) \, ds \le t m(\Gamma(f)) = 2t\mathcal{E}(f, f) < \infty,$$

Doob's inequality shows that $\tilde{M}^f$ (extended to all of $[0, \infty[$ by setting $\tilde{M}_t^f := \tilde{M}_{\zeta-}^f$ for $t \ge \zeta$) is a square integrable martingale under P_B^x for q.e. $x \in E$. Since $\mathcal{A}^B f \in L^2(m)$, the integral term in (3.25) is clearly continuous on $[0, \infty[$, $\mathbf{P}_B^m$-a.s. Also, $\lim_{t \uparrow \zeta} f(X_t^B) = 0$, $\mathbf{P}_B^m$-a.s. Taking this last assertion on faith for the moment, we see that

$$f(X_t^B) = f(X_0^B) + \int_0^t \mathcal{A}^B f(X_s^B) \, ds + \tilde{M}_t^f, \qquad \forall t \ge 0, \mathbf{P}_B^m\text{-a.s.} \tag{3.26}$$

Taking expectations of both sides of (3.26) we find that

$$P_t^B f = f + \int_0^t P_s^B \mathcal{A}^B f \, ds, \qquad m\text{-a.e.}, \forall t \ge 0.$$

From this it follows immediately that $t^{-1}[P_t^B f - f] \to \mathcal{A}^B f$ in $L^2(m)$ as $t \to 0$. Thus $f \in D(A^B)$ and $A^B f = \mathcal{A}^B f$, as desired.

To see that $\lim_{t \uparrow \zeta} f(X_t^B) = 0$, $\mathbf{P}_B^m$-a.s. on $\{\zeta < \infty\}$ we borrow an argument from [**Fi97**; Proof of (4.5)]. First notice that it suffices to show that $\liminf_{t \uparrow \zeta} |f(X_t^B)| = 0$, since the *existence* of the limit in question is guaranteed by the Martingale Convergence Theorem. Recall that $g \in \mathcal{C}$ and $gf = f$. Let $G := \{f \ne 0\}$, so that $1_G \le g$, and therefore

$$\mathbf{P}_B^m \int_0^t 1_G(X_s^B) \, d\langle M^B \rangle_s = \mathbf{P}_B^m \int_0^t g(X_s^B) \, d\langle M^B \rangle_s \le t \int_E g\beta^2 \, dm < \infty$$

because B is admissible and $m P_t^B \le m$ by part (a). Thus, $\int_0^t 1_G(X_s^B) \, d\langle M^B \rangle_s < \infty$ $\mathbf{P}_B^m$-a.s. for all $t > 0$. Fix $q \in bL^1(m)$ with $q > 0$ and set $v := U^1 q$. Then v is a strictly positive element of $\mathcal{D}$. Let τ_n denote the first exit time of X from the set $H_n := \{|f| + v > 1/n\}$. Then the sequence τ_n, $n \ge 1$, announces ζ with respect to $\mathbf{P}^m$. An argument on p. 15 of [**MZ85**] now shows that $\mathbf{P}_B^m$-a.s. on $\{\zeta < \infty\}$, either $\tau(H_n) < \zeta$ for all n, or $\langle M^B \rangle_{\zeta-} = \infty$. In the first case, $\lim_{t \uparrow \zeta} \left[|f(X_t^B)| + v(X_t^B) \right] = 0$. On the other hand, the finiteness of $\int_0^t 1_G(X_s^B) \, d\langle M^B \rangle_s$ forces $\liminf_{t \uparrow \zeta} 1_G(X_t^B) = 0$ $\mathbf{P}_B^m$-a.s. on $\{\zeta < \infty, \langle M^B \rangle_{\zeta-} = \infty\}$. But f vanishes on G^c, so $\liminf_{t \uparrow \zeta} |f(X_t^B)| = 0$ on $\{\zeta < \infty, \langle M^\ell \rangle_{\zeta-} = \infty\}$ as well. □

(3.27) Corollary. *Let B be an admissible drift and let the process X^B and its transition semigroup (P_t^B) be as in the statement of Theorem (3.17). Let $\hat{X}^B$ be the "subprocess" of X corresponding to the MF $\hat{L}^B$ defined by*

$$\hat{L}_t^B := L_t^{-B} \exp(-2\Lambda(M^B)_t) = L_t^B \circ r_t, \qquad 0 < t < \zeta. \tag{3.28}$$

Then the transition semigroup $(\hat{P}_t^B)$ of $\hat{X}^B$ is the adjoint semigroup of (P_t^B):

$$(\hat{P}_t^B f, g)_m = (f, P_t^B g)_m, \qquad \forall f, g \in L^2(m), \forall t > 0.$$

Proof. Using (3.28) as in the proof of (3.17)(a) we see that

$$\begin{aligned}
(f, P_t^B g)_m &= \mathbf{P}^m[f(X_0)\mathbf{P}^{X_0}(L_t^B g(X_t))] = \mathbf{P}^m[f(X_0) L_t^B g(X_t); t < \zeta] \\
&= \mathbf{P}^m[f(X_t) L_t^B \circ r_t g(X_0); t < \zeta] = \mathbf{P}^m[g(X_0)\hat{L}_t^B f(X_t)] \\
&= \mathbf{P}^m\left[g(X_0)\mathbf{P}^{X_0}[\hat{L}_t^B f(X_t)]\right] = \int_E g(x)\hat{P}_t^B f(x)\, m(dx) \\
&= (g, \hat{P}_t^B f)_m,
\end{aligned}$$

as claimed. ☐

Formally, the infinitesimal generator of $(\hat{P}_t^B)$ is an extension of $\hat{\mathcal{A}}^B f = \mathcal{A}f - B(f) + f \cdot \nu^B$, where ν^B is the Revuz measure of the PCAF $\Lambda(M^B)$. This can be made precise, at least when $\nu^B = q^B \cdot m$ and q^B is locally square integrable in the strong sense detailed in the following

(3.29) Theorem. *Let B be an admissible drift and suppose that the Revuz measure ν^B of the PCAF $\Lambda(M^B)$ takes the form $q^B \cdot m$, where $q^B \in L^2(g \cdot m)$ for all $g \in \mathcal{C}$. Then the L^2-infinitesimal generator of the semigroup $(\hat{P}_t^B)$ is an extension of*

$$\hat{\mathcal{A}}^B f := Af - B(f) - q^B \cdot f, \qquad f \in \mathcal{C}.$$

Proof. In view of the form of the MF $\hat{L}^B$, the process $\hat{X}^B$ can be constructed by applying a suitable Girsanov transformation to the symmetric process X^* obtained by "killing" X with the decreasing MF $\exp(-2\Lambda(M^B)_t)$. As is well known ([**FOT94**; §6.1] or [**Fi89**; §3]), X^* is symmetric with respect to m and has Dirichlet form $\mathcal{E}^*(u,v) = \mathcal{E}(uv) + \int_E uv\, q^B\, dm$, with domain $\mathcal{D} \cap L^2(q^B \cdot m)$. The process X^* need not be a diffusion in our sense—its lifetime has a non-trivial totally inaccessible part unless $q^B \cdot m = 0$. Nonetheless a modified form of (3.17) applies. To see what is needed, notice that although M^B can be viewed as a CAF of X^*, it is not a local martingale over X^*—the grim reaper requires compensation:

$$M_t^* := M_t^B + \int_0^t q^B(X_s^*)\, ds$$

is a local martingale (on $[0, \zeta^*[$) CAF of X^*. We now use the stochastic exponential $L_t^* := \exp(M_t^* - \frac{1}{2}\langle M^* \rangle_t)$ of M^* to perform a drift transformation of X^*. The resulting process is none other than $\hat{X}^B$, and the obvious modification of the proof of (3.17) leads to the asserted conclusion. We leave the details to the reader, noting only that (i) $\Lambda^*(M^*) = 0$ (with the obvious notation), so that (3.15)(b) is amply satisfied relative to X^*, and (ii) in the final part of the proof it must be shown that if $f \in \mathcal{C}$ then $\lim_{t\uparrow\zeta^*} f(X^*_t) = 0$, $\mathbf{P}^m_*$-a.s. on $\{\zeta^*_p < \infty\}$, where ζ^*_p is the predictable part of ζ^*. ∎

The following special case of (3.29) can also be deduced directly from (3.17), applied to B and $-B$. In view of the remarks made just prior to Theorem (3.17), it solves the problem raised at the end of section 1(d) of [**St96**].

(3.30) Corollary. *Let B be an admissible drift such that $\int_E B(f)\,dm = 0$ for all $f \in \mathcal{C}$. Let X^B (resp. $\hat{X}^B$) be the diffusion obtained from X via the Girsanov transformation based on the supermartingale MF $L^B_t = \exp(M^B_t - \frac{1}{2}\langle M^B\rangle_t)$ (resp. $\hat{L}^B_t = \exp(-M^B_t - \frac{1}{2}\langle M^B\rangle_t)$). Then the transition semigroups (P^B_t) and $(\hat{P}^B_t)$ of X^B and $\hat{X}^B$ are adjoint, as are the corresponding infinitesimal generators, which are extensions of $\mathcal{C} \ni f \mapsto Af + B(f)$ and $\mathcal{C} \ni f \mapsto Af - B(f)$ respectively.*

Proof. Simply notice that the condition $\int_E B(f)\,dm = 0$ for all $f \in \mathcal{C}$ implies that both $\Lambda(M^B)$ and $\Lambda(M^{-B}) = -\Lambda(M^B)$ are PCAFs, hence $\Lambda(M^B) = 0$. ∎

4. Examples

(4.1) Example. Let X be standard Brownian motion in a simply connected domain $D \subset \mathbf{R}^d$ and take $\mathcal{C}$ to be $C^\infty_c(D)$. The carré du champs operator associated with $\frac{1}{2}\Delta$ is $\Gamma(f) = |\nabla f|^2$, and there is no difficulty in checking that Hypothesis (2.5) is satisfied. Every derivation B is of the form $B(f) = \sum_{i=1}^d b_i \frac{\partial f}{\partial x_i}$, where $b := (b_1, \ldots, b_d)$ is a measurable mapping of D into $\mathbf{R}^d$. Consequently every derivation is Γ-bounded with (optimal) Γ-bound $\beta(x)$ equal to the Euclidean norm of $b(x)$, $x \in D$. A derivation B is admissible if and only if (i) the Euclidean norm $|b|$ is square integrable over each compact subset of D, and (ii) the distribution-sense divergence $\operatorname{div} b$ is a positive Radon measure on D.

(4.2) Example. The methods of this paper provide an alternative approach to the construction problem studied in [**Tr03a**] and [**Tr03b**]. To ease the exposition we restrict attention to the Brownian case, but note that the discussion carries over to the full context of those papers. Let K be a compact subset of $\mathbf{R}^d$ with interior G, and suppose that $\overline{G} = K$. Let $\mathcal{C} = C^\infty(K)$ denote the class of real-valued functions defined and smooth on a (variable) neighborhood of K. On $\mathcal{C}$ define the classical Dirichlet form

$$\mathcal{E}(u,v) := \frac{1}{2}\int_G \nabla u \cdot \nabla v\,dm, \qquad u, v \in \mathcal{C},$$

where m is normalized Lebesgue measure on G. The form $(\mathcal{E}, \mathcal{C})$ is closable, and its closure $(\mathcal{E}, \mathcal{D})$ is the Dirichlet form of the *modified reflected Brownian motion on* $\overline{G}$, as discussed in [**Fu97a**] and [**Fu97b**]. As in Example (4.1) the conditions of Hypothesis (2.5) are easily checked. Now let $b := (b_1, \ldots, b_d)$ be a measurable vector field on G such that $\int_G |b|^2\,dm < \infty$. The derivation $B(f) := \sum_{i=1}^d b_i \frac{\partial f}{\partial x_i}$ is admissible provided $\int_G B(f)\,dm = 0$ for all $f \in \mathcal{C}$, which we henceforth assume. Corollary (3.30) therefore applies, and [**Tr03b**; Thm. 2.3] follows as a special case, as do the results in [**Tr03b**; §3] because m is an excessive duality measure for X^B and X^{-B}.

(4.2) Example. Let E be a real separable Banach space and let m be a probability measure defined on the Borel subsets of E with $\operatorname{supp}(m) = E$. Let $H \hookrightarrow E$ be a real separable Hilbert space embedded densely and continuously in E. Writing E' for the (strong) dual of E and identifying H' with H, we have the dense continuous embedding $E' \hookrightarrow H$. Let $\mathcal{C}_0$ denote the class of functions on E of the form $f(\ell_1, \ell_2, \ldots, \ell_n)$, where $f \in C_b^\infty(\mathbf{R}^n)$, $\ell_i \in E'$ and $n \in \mathbf{N}$. For $u \in \mathcal{C}_0$ and $k \in E$ let $\frac{\partial u}{\partial k}$ denote the directional derivative $z \mapsto \lim_{s \to 0}[u(z + sk) - u(z)]/s$. A vector $k \in E$ is said to be *well m-admissible* if there exists $\gamma_k \in L^2(m)$ such that

$$\int_E \frac{\partial u}{\partial k}\, dm = -\int_E u\gamma_k\, dm \qquad \forall u \in \mathcal{C}_0. \tag{4.3}$$

Assume that there is a dense subspace $K \subset E'$ consisting of well m-admissible elements, and let $\mathcal{C}$ be defined as was $\mathcal{C}_0$ with the additional restriction that each ℓ_i is an element of K. For $u = f(\ell_1, \ldots, \ell_n) \in \mathcal{C}$ define

$$\mathcal{A}u = \sum_k \left[\frac{\partial}{\partial k}\left(\frac{\partial u}{\partial k}\right) + \gamma_k \frac{\partial u}{\partial k}\right], \tag{4.4}$$

where the sum runs over any orthonormal basis $K_0 \subset K$ of H containing $\ell_1, \ldots, \ell_n$. Clearly the carré du champs operator associated with $\mathcal{A}$ is given by

$$\Gamma(u, v) = \sum_k \left(\frac{\partial u}{\partial k}\right)\left(\frac{\partial v}{\partial k}\right), \tag{4.5}$$

and once it is noted that the right side of (4.10) is a finite sum, it is not hard to check that (2.6) is satisfied. Evidently,

$$\mathcal{E}(u, v) := (u, -\mathcal{A}v)_m = \tfrac{1}{2}\textstyle\int_E \Gamma(u, v)\, dm, \qquad u, v \in \mathcal{C}. \tag{4.6}$$

The form $(\mathcal{E}, \mathcal{C})$ is a closable symmetric Markovian form, and its closure $(\mathcal{E}, \mathcal{D})$ is a strongly local Dirichlet form. Notice that $1 \in \mathcal{C}$ and that $\mathcal{E}(1, 1) = 0$.

In the present situation, every derivation is of the form $\sum_{k \in K_0} b_k \frac{\partial u}{\partial k}$, where $\sum_{k \in K_0} b_k(x)^2 < \infty$ for all $x \in E$. Notice that this condition guarantees that the mapping $x \mapsto \sum_{k \in K_0} b_k(x) \cdot k$ is a measurable mapping of E into H. A derivation B is admissible if and only if (i) $\beta := |B|_H \in L^2(m)$ and (ii) $\sum_{k \in K_0}[b_k\gamma_k + (\partial b_k/\partial k)]$ is a positive Borel measure on E. Here the partial derivatives $\partial b_k/\partial k$ must be understood in the distribution sense.

Given an admissible derivation B, we have

$$\mathcal{A}^B u = \sum_k \left[\frac{\partial}{\partial k}\left(\frac{\partial u}{\partial k}\right) + \hat{\gamma}_k \frac{\partial u}{\partial k}\right] = \mathcal{A}u + B(u), \tag{4.7}$$

where

$$\hat{\gamma}_k = \gamma_k + b_k. \tag{4.8}$$

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