A Proof of Lehoczky's Theorem on Drawdowns

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1. Introduction. Let $X = (X_t)_{t\geq 0}$ be a regular diffusion process on an interval $E \subset \mathbb{R}$. Let $M_t := \max_{0\leq u\leq t} X_u$ denote the past maximum process of X, and for $\delta > 0$ define the "drawdown time" $\tau = \tau_\delta$ by

$$\tau = \inf\{t > 0 : X_t = M_t - \delta\}. \tag{1.1}$$

Our goal here is to use the excursion theory developed in [**F85**] to prove (the general form of) a theorem of J. Lehoczky concerning the joint distribution of τ and M_τ.

Before stating the result we introduce the necessary notation. Let $A = \inf E$, $B = \sup E$, and write $E^\circ =]A, B[$. We assume throughout the paper that $B \notin E$, and that $A \in E$ if and only if A is a regular boundary point which is not a trap for X. These assumptions imply that the transition kernels of X are absolutely continuous with respect to the speed measure m (recalled below). See §4.11 of [**IM74**].

The process X is realized as the coordinate process on the space Ω of paths $\omega: [0, +\infty[\to E \cup \{\Delta\}$ which are absorbed in the cemetery point $\Delta \notin E$ at time $\zeta(\omega)$, and which are continuous on $[0, \zeta(\omega)[$. The σ-fields $\mathcal{F}$ and $\mathcal{F}_t$ $(t \geq 0)$ are the usual Markovian completions of $\mathcal{F}^\circ = \sigma\{X_u : u \geq 0\}$ and $\mathcal{F}_t^\circ = \sigma\{X_u : 0 \leq u \leq t\}$ respectively. The law $\mathbf{P}^x$ on $(\Omega, \mathcal{F}^\circ)$ corresponds to X started at $x \in E$.

We assume that X admits no killing in E°, and let S and m denote the scale function and speed measure of X. The infinitesimal generator $\mathcal{G}$ of X has the form

$$\mathcal{G}u(x) \cdot m(dx) = du^+(x), \qquad x \in E^\circ, \tag{1.2}$$

for $u \in D(\mathcal{G})$, the domain of $\mathcal{G}$. Here and elsewhere u^+ denotes the (right-hand) scale derivative:

$$u^+(x) = \lim_{y\downarrow x} \frac{u(y) - u(x)}{S(y) - S(x)}. \tag{1.3}$$

Likewise, u^- denotes the left scale derivative.

As a regular diffusion, X admits local time; this is a jointly continuous (adapted) process $(L_t^y : t \geq 0, y \in E)$ normalized to be occupation density relative to m; that is

$$\int_0^t f(X_s)\, ds = \int_E f(x) L_t^x\, m(dx), \qquad \forall x \in E, t \geq 0, \tag{1.4}$$

for all bounded continuous f, almost surely. For a fixed level $y \in E$, the local time can be used to normalize the Itô excursion law [**I70**], for excursions from level y, as follows. Let $G(y)$ denote the (random) set of left-hand endpoints (in $]0, \zeta[$) of intervals contiguous to the level set $\{t > 0 : X_t = y\}$. Define the hitting time T_y by

$$T_y = \inf\{t > 0 : X_t = y\} \qquad (\inf \emptyset = +\infty). \tag{1.5}$$

The Itô excursion law n_y is determined by the identity

$$\mathbf{P}^x \sum_{u\in G(y)} Z_u\, F(\mathbf{e}^u) = \mathbf{P}^x \left(\int_0^\infty Z_u\, dL_u^y \right) \cdot n_y(F), \tag{1.6}$$

where $x \in E$, $\mathbf{e}^u$ is the excursion from y starting at time $u \in G$

$$\mathbf{e}_t^u = \begin{cases} X_{u+t}, & 0 \le t < T_y - u, \\ \Delta, & t \ge T_y - t. \end{cases} \tag{1.?}$$

$F \in p\mathcal{F}^\circ$, and $Z \ge 0$ is an $(\mathcal{F}_t)$-optional process. Under n_y the coordinate process $(X_t : t > 0)$ is strongly Markovian with semigroup (Q_t^y) given by

$$Q_t^y(x, f) = \mathbf{P}^x(f \circ X_t; t < T_y). \tag{1.7}$$

Finally, the point process of excursions below the maximum is defined as follows. For $t \ge 0$ set

$$\begin{aligned} H(\omega) &= \{u > 0 : X_u(\omega) = M_u(\omega)\}; \\ R_t(\omega) &= \inf\{u > 0 : u + t \in H(\omega)\}; \\ G(\omega) &= \{u > 0 : u < \zeta(w), R_{u-}(\omega) = 0 < R_u(\omega)\}. \end{aligned} \tag{1.8}$$

Thus G is the random set of left-hand endpoints of intervals contiguous to the random set H. As before, for each $u \in G$ we have an excursion $\mathbf{e}^u$ defined by

$$\mathbf{e}_t^u = \begin{cases} X_{u+t}, & 0 \le t < R_u, \\ \Delta, & t \ge R_u. \end{cases} \tag{1.9}$$

The point process $\Pi = \{(X_u, \mathbf{e}^u) : u \in G\}$ admits a Lévy system as detailed in the following proposition. In effect, the excursions below the running maximum, when indexed by their levels, form a Poisson point process in $E \times \Omega$, with intensity $(dy, d\omega) \mapsto dS(y) \cdot n_y^\downarrow(d\omega)$. Define a continuous increasing adapted process $C = (C_t : t \ge 0)$ by

$$C_t = \begin{cases} S(M_t) - S(M_0), & \text{if } t < \zeta, \\ C_{\zeta-} & \text{if } t \ge \zeta. \end{cases} \tag{1.10}$$

(1.11) Proposition. *For $Z \ge 0$ and $(\mathcal{F}_t)$-optional, and $F \in p\mathcal{F}^\circ$,*

$$\begin{aligned} \mathbf{P}^x \sum_{u\in G} Z_u\, F(\mathbf{e}^u) &= \mathbf{P}^x \int_0^\infty Z_u\, n_{X_u}^\downarrow(F)\, dC_u \\ &= \mathbf{P}^x \int_A^x Z_{T_y} 1_{\{T_y < +\infty\}} n_y^\downarrow(F)\, dS(y), \end{aligned} \tag{1.12}$$

where $n_y^\downarrow$ denotes the restriction of n_y to $\{\omega : \omega(t) < y, \forall t \in]0, \zeta(\omega)[\}$.

(1.13) Remark. The second equality in (1.12) follows from the first by the change of variable $u = T_y$.

We will also need (see §4.6 of [**IM74**]) the Laplace transform of T_y:

$$\mathbf{P}^x(e^{-\alpha T_y}) = \begin{cases} g_1^\alpha(x)/g_1^\alpha(y), & x \le y, \\ g_2^\alpha(x)/g_2^\alpha(y), & x \ge y, \end{cases} \tag{1.14}$$

where for fixed $\alpha > 0$, g_1^α and g_2^α are strictly positive, linearly independent solutions of

$$\mathcal{G}g(x) = \alpha g(x), \qquad x \in E^\circ; \tag{1.15}$$

g_1^α (resp. g_2^α) is an increasing (resp. decreasing) solution of (1.15) which also satisfies the appropriate boundary condition at A (resp. B). Both g_1^α and g_2^α are uniquely determined up to a positive multiple. As a rule we drop the superscript α, writing simply g_1 and g_2.

Define

$$b(z) := n_z^\downarrow\left(e^{-\alpha T_{z-\delta}}; T_{z-\delta} < \zeta\right) \tag{1.16}$$

and

$$c(z) := n_z^\downarrow\left(1 - e^{-\alpha\zeta}; T_{z-\delta} > \zeta\right). \tag{1.17}$$

(More explicit expression for b and c, in terms of the solutions g_1 and g_2, will be displayed below in the course of proving the Theorem.)

(1.18) Theorem. *The joint Laplace transform of M_τ and τ is given by*

$$\mathbf{P}^x[\exp(-\alpha\tau - \beta M_\tau)] = \int_x^B \exp\left(-\beta y - \int_x^y b(z)\,dS(z)\right) c(y)\,dS(y), \qquad x \in E. \tag{1.19}$$

In particular,

$$\mathbf{P}^x[M_\tau > y] = \exp\left(-\int_x^y \frac{dS(z)}{S(z) - S(z-\delta)}\right), \qquad x \le y < B. \tag{1.20}$$

2. Proofs.

(2.1) Lemma. *For $\alpha > 0$ and $\delta > 0$,*

$$n_y^\downarrow\left(e^{-T_{y-\delta}}; T_{y-\delta} < \zeta\right) = \frac{g_1^-(y)g_2(y) - g_1(y)g_2^-(y)}{g_1(y)g_2(y-\delta) - g_1(y-\delta)g_2(y)}. \tag{2.2}$$

Proof. By a result apparently due originally to [**DS53**] (see also [**IMK**; §4.10]), for $y-\delta < y-\epsilon < y$, we have

$$\mathbf{P}^{y-\epsilon}\left(e^{-\alpha T_{y-\delta}}; T_{y-\delta} < T_y\right) = \frac{g_1(y)g_2(y-\epsilon) - g_1(y-\epsilon)g_2(y)}{g_1(y)g_2(y-\delta) - g_1(y-\delta)g_2(y)}. \tag{2.3}$$

Subtract and add $g_1(y)g_2(y)$ in the numerator on the right, and then divide both sides by $S(y) - S(y-\epsilon)$ and send $\epsilon \downarrow 0$. The limit on the left is (as is well known, and easily deduced) $n_y^\downarrow(\exp(-\alpha T_{y-\delta}); T_{y-\delta} < \zeta)$. The left-hand scale derivatives of g_1 and g_2 exist because both functions are in the domain of the generator, so the limit of the right side is as indicated. ∎

Because g_1 and g_2 are linearly independent, any other linearly independent pair of solutions of (1.15) may be used in (2.2) instead, and likewise in (2.5) below.

Essentially the same computation yields

(2.4) Lemma.

$$n_y^{\downarrow}\left(1-e^{-\alpha\zeta}; T_{y-\delta}>\zeta\right) = \frac{g_1^-(y)g_2(y-\delta)-g_1(y-\delta)g_2^-(y)}{g_1(y)g_2(y-\delta)-g_1(y-\delta)g_2(y)}. \tag{2.5}$$

The formulas presented in these two lemmas are related to the spectral decomposition of the Laplace transform of the hitting times of X that is treated in [**K82**].

Proof of the Theorem. We start with (1.20). Observe that when $X_0 = x$, we have $\{M_\tau > y\} = \{N_{x,y} = 0\}$, where

$$N_{x,y} := \{(z,\mathbf{e}) \in \Pi : x < z \le y, T_{z-\delta}(\mathbf{e}) < \zeta(\mathbf{e})\}. \tag{2.6}$$

By Proposition (1.11), under $\mathbf{P}^x$ the random variable $N_{x,y}$ has the Poisson distribution with mean value

$$\int_{]x,y]} n_z^{\downarrow}[T_{z-\delta} < \zeta]\, dS(z). \tag{2.7}$$

Notice that the $\alpha = 0$ version of (2.3) is the well known

$$\mathbf{P}^{z-\epsilon}(T_{z-\delta} < T_z) = \frac{S(z)-S(z-\epsilon)}{S(z)-S(z-\delta)} \tag{2.8}$$

and so

$$n_z^{\downarrow}(T_{z-\delta} < \zeta) = \frac{1}{S(z)-S(z-\delta)}, \tag{2.9}$$

as in the proof of (2.2). Formula (1.20) now follows from a well-known property of the Poisson distribution.

We now turn to the proof of (1.19). Taking a cue from [**L77**], we compute

$$\mathbf{P}^x[e^{-\alpha\tau} \mid M_\tau = y] \tag{2.10}$$

for $y > x$. Notice that, given that $M_\tau = y$, τ is the sum of the "run-up time" to level y

$$\rho := \sum_{(z,\mathbf{e})\in\Pi} \zeta(\mathbf{e}) \cdot 1_{x<z\le y, T_{z-\delta}(\mathbf{e})>\zeta(\mathbf{e})} \tag{2.11}$$

(because the Lebesgue measure of H is 0, a.s.) and a random variable σ that is independent of ρ and which has the same distribution as $T_{y-\delta}$ under $n_y^{\downarrow}(\,\cdot\,|T_{y-\delta} < \zeta)$. By the Poisson master formula [**K02**; (3.6)]

$$\mathbf{P}^x[\exp(-\alpha\rho)] = \exp\left(-\int_{]x,y]} n_z^{\downarrow}(1-e^{-\alpha\zeta})\, dS(z)\right) \tag{2.12}$$

Combining this with Lemmas (2.1) and (2.4), we obtain

$$
\begin{aligned}
\mathbf{P}^x[e^{-\alpha\tau} \mid M_\tau = y] &= \exp\left(-\int_{]x,y]} n_z^{\downarrow}(1-e^{-\alpha\zeta}, T_{z-\delta} > \zeta)\, dS(z)\right) \cdot n_y^{\downarrow}(e^{-\alpha T_{y-\delta}} | T_{y-\delta} < \zeta) \\
&= \exp\left(-\int_x^y c(z)\, dS(z)\right) \cdot b(y) \cdot (S(y) - S(y-\delta)).
\end{aligned}
\tag{2.13}
$$

In view of (1.20), which shows that

$$
\mathbf{P}^x[M_\tau \in dy] = \frac{dS(y)}{S(y) - S(y-\delta)}, \qquad y > x,
$$

the Theorem is proved. □

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