A Proof of the Jensen-Dragomir Inequality

P.J. Fitzsimmons

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A general (measure theoretic) version of Dragomir's gloss on Jensen's inequality has recently been published by Aldaz [**1**]. Below we state this theorem for random variables in d-dimensional Euclidean space, and provide what we feel is a simple proof. To state the theorem let μ and ν be Borel probability measures concentrated on some open set $U \subset \mathbf{R}^d$ and let $\varphi : U \to \mathbf{R}$ be convex. Assume that $\int_U |x|\,\mu(dx) + \int_U |x|\,\nu(dx) < \infty$. Let $V(\mu) := \int_U \varphi(x)\,\mu(dx) - \varphi(\int_U x\,\mu(dx))$ denote the "Jensen functional" associated with φ. Observe that $\int_U \varphi(x)\,\mu(dx)$ is well-defined, taking values in $(-\infty, +\infty]$.

Theorem. *Suppose that $\mu \ll \nu$ and let $k \in (0,\infty]$ denote the $L^\infty(\nu)$-norm of the Radon-Nikodym derivative $h := d\mu/d\nu$. Then*

$$V(\mu) \le k \cdot V(\nu). \tag{1}$$

Proof. Given $y \in U$ let $\gamma(y) \in \mathbf{R}^d$ be a subgradient of φ at y. This means that

$$f(x,y) := \varphi(x) - \varphi(y) - \gamma(y)\cdot(x-y) \ge 0$$

for all $x \in U$. We write $m = \int_U x\,\mu(dx)$ and $n = \int_U x\,\nu(dx)$, and compute

$$\begin{aligned}
V(\mu) &= \int_U [\varphi(x) - \varphi(m)]\,\mu(dx) = \int_U f(x,m)\,\mu(dx)\\
&= \int_U f(x,n)\,\mu(dx) + \int_U [f(x,m) - f(x,n)]\,\mu(dx)\\
&\le k \cdot \int_U f(x,n)\,\nu(dx) + \int_U [\varphi(n) - \varphi(m) - \gamma(m)\cdot(x-m) + \gamma(n)\cdot(x-n)]\,\mu(dx)\\
&= k\cdot V(\nu) + \int_U [\varphi(n) - \varphi(m) + \gamma(n)\cdot(x-n)]\,\mu(dx)\\
&= k\cdot V(\nu) - f(m,n)\\
&\le k \cdot V(\nu).
\end{aligned}$$

(We adhere to the usual convention that $\infty \cdot 0 = 0$; in particular, $k \cdot \int_U f(x,n)\,\nu(dx) = 0$ if $k = \infty$ and $\int_U f(x,n)\,\nu(dx) = 0$.) $\square$

There is a companion lower bound $k_* V(\nu) \le V(\mu)$ in which k_* is the ν-essential infimum of h; this bound is obtained by reversing the roles of μ and ν. See [**1**; Cor. 2.9] for details.

Henceforth we assume that φ is strictly convex. Suppose that the two sides of (1) are equal (and finite). If they are both equal to 0, then by the aforementioned convention, $V(\mu) = 0 = V(\nu)$, so (by strictly convexity of φ) both μ and ν are unit point masses at $m = n$.

If the two sides of (1) are equal and strictly positive (as well as finite) then k must be finite, both $\int_U \varphi(x)\,\mu(dx)$ and $\int_U \varphi(x)\,\nu(dx)$ must be finite, and

$$f(m, n) = 0, \tag{2}$$

and

$$\int_U [k - h(x)] \cdot f(x, n)\,\nu(dx) = 0. \tag{3}$$

Because φ is strictly convex, (2) forces $m = n$, and then (3) implies that

$$\nu\{x \in U : h(x) = k \text{ or } x = m\} = 1. \tag{4}$$

Define $p := \nu\{m\}$ and $\widetilde{\nu}(A) := (1-p)^{-1}\nu(A \setminus \{m\})$ for Borel $A \subset U$. (If $p = 1$ then we take $\widetilde{\nu}$ to be ϵ_m, the unit point mass at m.) Then ν decomposes as

$$\nu = p \cdot \epsilon_m + (1-p)\widetilde{\nu},$$

and, in view of (4),

$$\mu = ph(m) \cdot \epsilon_m + (1-p)k \cdot \tilde{\nu}.$$

(Notice that $h(m)$ is uniquely determined as $\mu\{m\}/p$ if $p > 0$.) Consequently, $1 - \mu\{m\} = k \cdot (1 - \nu\{m\})$.

References

[1] Aldaz, J. M.: A measure-theoretic version of the Dragomir-Jensen inequlity. *Proc. Amer. Math. Soc.* **140** (2012) 2391–2399.

[2] Dragomir, S.S.: Bounds for the normalised Jensen functional. *Bull. Australian. Math. Soc.* **74** (2006) 471–478.