DIRICHLET FUNCTIONS

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Abstract. We provide a short proof that if X is a symmetric diffusion and $u(X_t)$ is a Dirichlet process (basically, a process of finite quadratic variation) then u is locally in the Dirichlet space of X.

The following result was proved (in a global form) for elliptic divergence form diffusions in $\mathbf{R}^d$ by Lyons & Zheng [LZa], and (in a local form) by Chitashvili & Mania [CM] for one-dimensional Brownian motion. See also[EW]. The proof presented here covers a very general case, and seems to be simpler that that used in the earlier work cited. It is based on the idea of even and odd additive functionals as developed in [FITa] and [FITb]. See also [FPS].

The context is a symmetric Markov diffusion process $X = (X_t)_{t\geq 0}$ in a general state space E.

Theorem. *Let $u : E \to \mathbf{R}$ be a Borel function such that $A_t^u := \tilde{u}(X_t) - \tilde{u}(X_0)$ is a Dirichlet process; that is, decomposable as $M_t + N_t$ where M is continuous local martingale and N is locally of zero energy. Then $u \in \mathcal{D}_{\text{loc}}$.*

Proof. The decomposition cited is necessarily unique. This uniqueness and the additivity of A^u imply that M and N are CAFs of X. But we know from [FITa] that a CAF locally of zero energy is necessarily even. Thus, A^u is the odd part of M. Lemma 3.15 of [FITb] now implies that $u \in \mathcal{D}_{\text{loc}}$. $\square$

Proceeding somewhat speculatively, let A be an arbitrary CAF. By a result of Oshima and Yamada [OY] we have

$$A_t = f(X_t) - f(X_0) + M_t,$$

where f is quasi-continuous and M is a local martingale CAF. If A is even, then it is the even part of M, namely $-\Lambda(M)$, so $f \in \mathcal{D}_{\text{loc}}$ and $M = -M^f$ in this case. [The linear functional $M \mapsto \Lambda(M)$, from martingale CAFs to zero-energy CAFs, was introduced by S. Nakao in [NAK], and further studied in [FITa]; Nakao uses the notation $\Gamma(M)$ rather than $\Lambda(M)$.] If A is odd, then it coincides with

$$A^f + M + \Lambda(M).$$

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In general,

$$A = [-\Lambda(M)] + [A^f + M + \Lambda(M)]$$

is the decomposition of A into even and odd parts. Of course A has finite quadratic variation if and only if A^f does.

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