On a result of Davis and Suh

Let B be a standard real-valued Brownian motion, and define $S_t := \sup_{0\le s\le t} |B_s|$. Davis and Suh give a proof in [**1**] of the following result, in which

$$Y_t = S_t^{p-2}[B_t^2 - t] + cS_t^p, t \ge 0$$

and $c_0 := (2-p)/p$.

Theorem.

(i) Suppose $0 < p \le 2$. Then Y is a submartingale if and only if $c \ge c_0$.

(ii) Supose $p \ge 2$. Then Y is a supermartingale if and only if $c \le c_0$.

The proof that follows seems to be a little simpler than the one offered in [**1**].

Proof. Let M denote the martingale $B_t^2 - t$. The case $p = 2$ is trivial, hence excluded. By Itô's formula

$$Y_t = \int_0^t S_u^{p-2}\, dM_u + (p-2)\int_0^t \left[M_u + \frac{cp}{p-2}S_u^2\right] S^{p-3}\, dS_u. \tag{1}$$

The first term on the right side of (1) is a martingale; the second is continuous and of integrable variation. It remains only to show that it is increasing in case (i) and decreasing in case (ii).

(i) In this case we rewrite the second term on the right side of (1) as

$$(2-p)\int_0^t \left[\frac{cp}{2-p}S_u^2 - M_u\right] S^{p-3}\, dS_u. \tag{2}$$

By hypothesis, $cp/(2-p) \ge 1$, and clearly $M_u \le S_u^2$ for all $u \ge 0$, so the integrand in (2) is non-negative, as desired.

(ii) In this case the condition $c \ge c_0$ amounts to the statement that $cp/(p-2) \le -1$. Thus,

$$M_u + \frac{cp}{p-2}S_u^2 \le M_u - S_u^2 = B_u^2 - u - S_u^2 \le -u \le 0.$$

□

References

[1] Davis, B. and Suh, J.: On Burkholder's supermartingales, *Illinois. J. Math.* **50** (2006) 313–322.