Cramér in Continuous Time

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Theorem. *Let* $(B_t)_{t\ge 0}$ *be a standard 1-dimensional Brownian motion defined on a filtered probability space* $(\Omega, \mathcal{F}, \mathcal{F}_t, P)$*, with* $B_0 = 0$*. Assume that* $(\mathcal{F}_t)$ *is the augmented natural filtration of* (B_t)*. Let* M *and* N *be independent* $(\mathcal{F}_t)$*-local martingales such that* $M + N = B$*. Then there is a Borel set* $A \subset [0,\infty[$ *such that*

$$M_t = \int_0^t 1_A(s)\, dB_s, \qquad N_t = \int_0^t 1_{A^c}(s)\, dB_s, \qquad \forall t \ge 0, \text{ a.s. } P.$$

Proof. Observe that $M_0 = N_0 = 0$, owing to the independence of M and N. Also, it follows easily from Cramer's theorem that the trivariate process (M, N, B) is (jointly) Gaussian. In particular, M and N are actually martingales. But a Gaussian martingale must have independent increments. More precisely, $(M_t - M_s)_{t\ge s}$ is independent of $\mathcal{F}_s$ for each fixed $s > 0$, and similarly for N. It follows from a theorem of Lévy that there are (independent) Brownian motions α and β and (deterministic) continuous increasing functions a and b (mapping $[0,\infty[$ into itself) such that

$$M_t = \alpha_{a(t)}, \qquad N_t = \beta_{b(t)}, \qquad \forall t \ge 0, \text{ a.s. } P.$$

On the other hand, by the predictable representation property of Brownian motion, there are predictable processes H and K such that

$$\int_0^t (H_s^2 + K_s^2)\, ds = t$$

and

$$M_t = \int_0^t H_s\, dB_s \qquad N_t = \int_0^t K_s\, dB_s,$$

for all $t > 0$, a.s. P. The following properties are evident:

$$H_s(\omega) + K_s(\omega) = 1, \qquad \text{for Leb} \times P \text{ a.e. } (s,\omega) \in [0,\infty[\times\Omega; \tag{1}$$

$$H_s(\omega) \cdot K_s(\omega) = 0, \qquad \text{for Leb} \times P \text{ a.e. } (s,\omega) \in [0,\infty[\times\Omega; \tag{2}$$

$$\int_0^t H_s^2\, ds = \langle M \rangle_t = a(t), \qquad \forall t \ge 0, \text{ a.s. } P. \tag{3}$$

It follows from (1) and (2) that both H and K take values in $\{0,1\}$ for Leb $\times P$ a.e. $(s,\omega) \in [0,\infty[\times\Omega$. Now (3) implies that a is absolutely continuous on compact time intervals and that $a'(s) = H_s(\omega)$ for Leb $\times P$ a.e. $(s,\omega) \in [0,\infty[\times\Omega$. Consequently, $H_s(\omega) = 1_A(s)$ for Leb $\times P$ a.e. $(s,\omega) \in [0,\infty[\times\Omega$, where $A := \{s : a'(s) > 0\}$. Similarly, $K = 1_{A^c}$. The proof is complete. ∎