Math 294, Spring 2024

Homework 3 — Due May 22

In problems 1 and 2, $(W_t)_{t\geq 0}$ is a (standard) Brownian motion defined on some filtered probability space $(\Omega, \mathcal{F}, (\mathcal{F}_t), \mathbf{P})$.

1. Show that $X_t = W_t^3 - 3tW_t$ is a martingale. [Hint: One way to do this is to compute the conditional expectation $\mathbf{E}[X_t|\mathcal{F}_s]$ $(0 < s < t)$ by using the "trick" of writing $W_t = W_s + (W_t - W_s)$. Another is to use Itô's formula to show that $X_t = 3\int_0^t (W_s^2 - s)\, dW_s$.]

2. Fix $\sigma > 0$ and $\mu \in \mathbf{R}$. Express $X_t := \exp(\sigma W_t + \mu t)$ as an Itô process; that is, find Y and b such that $X_t = X_0 + \int_0^t Y_s\, dW_s + \int_0^t b_s\, ds$. Under what condition on σ and μ is X a martingale?

3. Section 4.10, p. 86, Exercise 1.

4. Section 4.10, p. 86, Exercise 2, parts (a) and (b) only.