Math 294, Spring 2024

Homework 2 — Due May 1

1. Section 2.4, Exercise 6 (Don't do part (d).)

2. Section 3.7, Exercise 1

3. Section 3.7, Exercise 2

4. Section 3.7, Exercise 4

5. Let $(\Omega, \mathcal{F}, \mathbf{P})$ be a probability space with filtration $\{\mathcal{F}_t : t = 0, 1, 2, \ldots, T\}$. Assume that Ω is finite. An adapted sequence $\{X_t : t = 0, 1, \ldots, T\}$ is a *submartingale* provided $\mathbf{E}[X_t|\mathcal{F}_{t-1}] \geq X_{t-1}$ for $t = 1, 2, \ldots, T$. In this case $\mathbf{E}[X_t|\mathcal{F}_s] \geq X_s$ for all $0 \leq s < t \leq T$. For example, if $\{M_t : t = 0, 1, \ldots, T\}$ is a martingale, then $X_t := |M_t|$ is a (non-negative) submartingale, by Jensen's inequality.

(a) Let $\tau : \Omega \to \{0, 1, \ldots, T\}$ be a stopping time. Let $\{X_t : t = 0, 1, \ldots, T\}$ be a non-negative submartingale with $X_0 = 0$. Explain why

$$\mathbf{E}[X_t; \tau = t] \leq \mathbf{E}[X_T; \tau = t], \qquad \forall t = 1, 2, \ldots, T.$$

Deduce from this that $\mathbf{E}[X_\tau] \leq \mathbf{E}[X_T]$.

(b) Let $\{X_t : t = 0, 1, \ldots, T\}$ be as in part (a), and define $\overline{X} := \max_{1 \leq t \leq T} X_t$. Fix $b > 0$. Explain why

$$b \cdot 1_{\{\overline{X} > b\}} \leq X_\tau,$$

where

$$\tau := \min\{t \in \{1, 2, \ldots, T\} : X_t > b\} \wedge T.$$

(c) Using (a) and (b) deduce that

$$\mathbf{P}\left[\max_{1 \leq t \leq T} X_t > b\right] \leq \frac{1}{b} \cdot \mathbf{E}[X_T], \qquad b > 0.$$

(This is Doob's maximal inequality.)