In the problems below, \( X = (X_n)_{n=0}^\infty \) is a Markov chain with countable state space \( S \) and transition probability matrix \( P = \{p(x, y)\}_{x, y \in S} \). We suppose that \( X \) has been constructed on the sequence space \( \Omega = E^{\{0,1,2,\ldots\}} \), and that \( \mathbb{P}_x \) is the probability measure on \( \Omega \) corresponding to the initial condition \( X_0 = x \). \( \mathbb{N}_0 \) denotes the non-negative integers \( \{0, 1, 2, \ldots\} \). Other notation is that used in class.

Problems (or parts of problems) marked with a * should be attempted, but need not be handed in.

1. Consider a function \( h : S \to [0, \infty) \) and a real number \( \alpha > 0 \) such that \( \sum_{y \in S} p(x, y)h(y) = \alpha \cdot h(x) \) for all \( x \in S \) (briefly, \( Ph = \alpha h \)). Equivalently, \( \mathbb{E}_x[h(X_1)] = \alpha \cdot h(x) \) for all \( x \in S \). (Such a function might be called \( \alpha \)-invariant.)

   (a) Show that the sequence \( \alpha^{-n}h(X_n), n \geq 0 \), is a martingale (under \( \mathbb{P}_\mu \) for any initial distribution \( \mu \) with \( \mu \cdot h = \sum_{x \in S} \mu(x)h(x) < \infty \)).

   (b) Suppose now that each state of \( S \) is recurrent. Show that if \( 0 < \alpha < 1 \), then \( h(x) = 0 \) for all \( x \in S \). [Hint: Use the Martingale Convergence Theorem, and consider the sequence of numbers

   \[ h(X_0(\omega)), h(X_1(\omega))/\alpha, h(X_2(\omega))/\alpha^2, \ldots \]

   for a “good” \( \omega \).]

2. Space-time invariant functions. The tail \( \sigma \)-field of \( X \) is

   \[ \mathcal{T} = \cap_n \theta_n^{-1}(\mathcal{F}). \]

Let \( Z \) be a bounded \( \mathcal{T} \)-measurable random variable. Then for each positive integer \( n \) there exists a unique bounded random variable \( Z_n \) such that \( Z = Z_n \circ \theta_n \). [This uniqueness guarantees that \( Z_{n+1} \circ \theta_1 = Z_n \) for all \( n \in \mathbb{N} \), because either of these random variables when composed with \( \theta_n \) yields \( Z \).] Define \( h(x, n) := \mathbb{E}_x[Z_n] \).

   (a) Show that \( h \) is a space-time invariant function, in the sense that

   \[ \sum_y p(x, y)h(y, n + 1) = h(x, n) \]

for all \( x \in S \) and all \( n \in \mathbb{N}_0 \). Deduce that \( h(X_n, n + m) \) is a \( \mathbb{P}_\mu \)-martingale, for every initial distribution \( \mu \) and every \( m \in \mathbb{N}_0 \).

   (b) Conversely, suppose that \( g(x, n) \) is a bounded function on \( S \times \mathbb{N}_0 \) such that

   \[ \sum_y p(x, y)g(y, n + 1) = g(x, n) \]

for all \( x \in S \) and all \( n \in \mathbb{N}_0 \). By part (a), \( g(X_n, n + m) \) is a \( \mathbb{P}_\mu \)-martingale for every initial distribution \( \mu \) and every \( m \in \mathbb{N}_0 \). Define \( Y_m := \liminf_{n \to \infty} g(X_n, n + m) \), a \( \mathcal{T} \)-measurable random
variable for each \( m \in \mathbb{N}_0 \). Show that if we set \( Y := Y_0 \), then \( Y = Y_m \circ \theta_m \) and \( g(x, m) = \mathbb{E}_x[Y_m] \), for each \( x \in S \) and each \( n \in \mathbb{N}_0 \).

*(c) Parts (a) and (b) establish a one-to-one correspondence between the class of bounded \( T \)-measurable functions and the class of bounded space-time invariant functions. Use this correspondence to prove that \( T \) is trivial (i.e., that \( P_\mu(A) = 0 \) or \( 1 \), for all \( A \in T \) and all initial distributions \( \mu \)) if and only if all bounded space-time invariant functions are constant.

3. A knight moves on a \( 8 \times 8 \) “chessboard,” at each step choosing randomly from his legal moves. What is the expected number of moves required for the knight to return to his original position, assuming he starts in a corner? [A knight’s move is L-shaped: one step in one direction followed by two steps in a perpendicular direction.]

4. Assume that \( X \) is irreducible. Suppose there is a finite set \( F \subset S \), a function \( w : S \to [0, \infty) \) with \( P_w(x) < \infty \) for all \( x \in F \), and a constant \( b > 0 \) such that \( P_w(x) \leq w(x) - b \) for all \( x \not\in F \). Define \( D := \inf\{n \geq 0 : X_n \in F \} \).

(a) Show that \( w(X_{n \wedge D}) + b \cdot (n \wedge D) \) is a non-negative supermartingale, under \( P_x \) for each \( x \in S \).

(b) Deduce from (a) that \( \mathbb{E}_x[D] \leq w(x)/b \) for all \( x \in S \).

(c) Deduce from (b) that \( X \) is positive recurrent.