Math 280C, Spring 2005

Girsanov's Theorem

In what follows, $(\Omega, \mathcal{F}, \mathbf{P})$ is the canonical sample space of the Brownian motion $(B_t)_{t\ge 0}$ with $B_0 = 0$; other notation is that used in class.

Let $\mathbf{Q}$ be a second probability measure on $(\Omega, \mathcal{F})$ that is locally mutually absolutely continuous with respect to $\mathbf{P}$ (in symbols $\mathbf{Q} \sim \mathbf{P}$), in the sense that that $\mathbf{Q}(A) = 0$ if and only if $\mathbf{P}(A) = 0$, for all $A \in \mathcal{F}_t$ and all $t > 0$. Then the restrictions $\mathbf{Q}^{(t)}$ and $\mathbf{P}^{(t)}$, of $\mathbf{Q}$ and $\mathbf{P}$ to $\mathcal{F}_t$, are likewise mutually absolutely continuous, and the process of Radon-Nikodym derivatives

$$Z_t := \frac{dQ^{(t)}}{dP^{(t)}} \qquad t \ge 0,$$

is a $\mathbf{P}$ martingale. We can (and do) choose a modification of Z that is right continuous. In fact, by the martingale representation theorem, the process Z has continuous paths. Notice that $\mathbf{E}^{\mathbf{P}}[Z_0] = 1$, while Blumenthal's zero-one law implies that $\mathcal{F}_{0+}$ is $\mathbf{P}$-trivial, hence $\mathbf{Q}$-trivial. Therefore $\mathbf{P}[Z_0 = 1] = 1 = \mathbf{Q}[Z_0 = 1]$.

Just as in discrete time, a positive martingale "sticks" at zero once it attains that value. But $\mathbf{Q}[Z_t = 0] = \int_{\{Z_t=0\}} Z_t \, d\mathbf{P} = 0$, so $Z_t > 0$, $\mathbf{Q}$-a.s., hence $\mathbf{P}$-a.s. It follows from this and path continuity that

$$\inf_{0\le s\le t} Z_s(\omega) > 0, \qquad \forall t > 0, \text{ for } \mathbf{P}\text{-a.e. } \omega.$$

This implies that the stochastic integral

$$M_t := \int_0^t Z_s^{-1} \, dZ_s, \qquad t \ge 0,$$

exists and defines a continuous local martingale $(M_t)_{t\ge 0}$. By the martingale representation theorem, there exists $H \in \mathcal{L}^2_{\mathrm{loc}}$ such that $M = H \bullet B$. We then have,

$$Z_t = 1 + \int_0^t Z_s \, dM_s = 1 + \int_0^t Z_s H_s \, dB_s, \qquad t \ge 0,$$

so by the uniqueness of the stochastic exponential as solution of such an equation,

$$Z_t = \exp\left(M_t - \frac{1}{2}\langle M\rangle_t\right), \qquad \forall t \ge 0, \mathbf{P}\text{-a.s.}$$

Now suppose that $F \in b\mathcal{F}_s$ and $G \in \mathcal{F}_t$, where $0 \le s < t$. Then

$$\begin{aligned}
\mathbf{E}^{\mathbf{Q}}[FG] &= \mathbf{E}^{\mathbf{P}}[FGZ_t] = \mathbf{E}^{\mathbf{P}}\left[F\mathbf{E}^{\mathbf{P}}[GZ_t|\mathcal{F}_s]\right] \\
&= \mathbf{E}^{\mathbf{P}}\left[F\frac{\mathbf{E}^{\mathbf{P}}[GZ_t|\mathcal{F}_s]}{\mathbf{E}^{\mathbf{P}}[Z_t|\mathcal{F}_s]} Z_s\right] \\
&= \mathbf{E}^{\mathbf{Q}}\left[F\frac{\mathbf{E}^{\mathbf{P}}[GZ_t|\mathcal{F}_s]}{\mathbf{E}^{\mathbf{P}}[Z_t|\mathcal{F}_s]}\right].
\end{aligned}$$

It follows that $\mathbf{Q}$ conditional expectations can be expressed in terms of $\mathbf{P}$ conditional expectations by the formula

$$\mathbf{E}^{\mathbf{Q}}[G|\mathcal{F}_s] = \frac{\mathbf{E}^{\mathbf{P}}[GZ_t|\mathcal{F}_s]}{\mathbf{E}^{\mathbf{P}}[Z_t|\mathcal{F}_s]} = \frac{\mathbf{E}^{\mathbf{P}}[GZ_t|\mathcal{F}_s]}{Z_s}, \qquad G \in b\mathcal{F}_t, 0 \le s < t. \tag{1}$$

Next let $N = (N_t)_{t\ge 0}$ be an adapted process. Then

$$\mathbf{E}^{\mathbf{Q}}\left[|N_t|\right] = \mathbf{E}^{\mathbf{P}}\left[|N_t|Z_t\right],$$

so N_t is $\mathbf{Q}$-integrable if and only if N_tZ_t is $\mathbf{P}$-integrable. If things are so, then by (1)

$$\mathbf{E}^{\mathbf{Q}}[N_t|\mathcal{F}_s] = \frac{\mathbf{E}^{\mathbf{P}}[N_tZ_t|\mathcal{F}_s]}{Z_s},$$

which is equal to N_s (a.s.) if and only if $\mathbf{E}^{\mathbf{P}}[N_tZ_t|\mathcal{F}_s] = N_sZ_s$. In other words, N is a $\mathbf{Q}$-martingale if and only if NZ is a $\mathbf{P}$-martingale. In particular, every right continuous $\mathbf{Q}$-martingale is, in fact, continuous.

Consider now an Itô process

$$X_t = X_0 + \int_0^t K_s\, dB_s + \int_0^t v_s\, ds = Y_t + V_t.$$

Our aim is to describe in concrete terms the circumstances under which X is a $\mathbf{Q}$ local martingale. By the preceding paragraph and a localization argument, X is a $\mathbf{Q}$ local martingale if and only if XZ is a $\mathbf{P}$ local martingale. But, by Itô's formula,

$$\begin{aligned} X_tZ_t &= X_0 + \int_0^t X_s\, dZ_s + \int_0^t Z_s\, dX_s + \langle X, Z\rangle_t \\ &= X_0 + \int_0^t X_s\, dZ_s + \int_0^t Z_sK_s\, dB_s + \int_0^t Z_sv_s\, ds + \int_0^t K_sH_sZ_s\, ds \\ &= X_0 + \int_0^t X_s\, dZ_s + \int_0^t Z_sK_s\, dB_s + \int_0^t (v_s + K_sH_s)Z_s\, ds. \end{aligned}$$

It follows that XZ is a $\mathbf{P}$ local martingale if and only if $(v_s + K_sH_s)Z_s = 0$ for a.e. $(\omega, s) \in \Omega \times [0, \infty)$. But $Z > 0$, so this condition holds if and only if $v = -HK$ a.e. on $\Omega \times [0, \infty)$. Another way of saying this is that for any $K \in \mathcal{L}^2_{\text{loc}}$, the process

$$X_t = Y_t - \langle Y, M\rangle_t$$

is a $\mathbf{Q}$ local martingale, where $Y = K \bullet B$ and $M = H \bullet B$ are $\mathbf{P}$ local martingales, as above. The moral of the story is that when we change the governing measure from $\mathbf{P}$ to $\mathbf{Q}$ we must compensate by subtracting the term $\langle Y, M\rangle$ to turn the $\mathbf{P}$ local martingale Y into a $\mathbf{Q}$-local martingale.

The above discussion can be summarized as follows.

Cameron-Martin-Girsanov Theorem. *(a) If* $\mathbf{Q}$ *is a probability measure on* $(\Omega, \mathcal{F})$ *that is locally mutually absolutely continuous with respect to* $\mathbf{P}$*, then there is a local martingale* $M = H \bullet B$ $(H \in \mathcal{L}^2_{\text{loc}})$ *such that the Radon-Nikodym martingale* $Z_t := d\mathbf{Q}^{(t)}/d\mathbf{P}^{(t)}$ *is equal to the exponential martingale* $\exp(M_t - \frac{1}{2}\langle M \rangle_t)$*. Moreover, if* N *is a* $\mathbf{P}$ *local martingale, then* $\tilde{N} := N - \langle N, M \rangle$ *is a* $\mathbf{Q}$ *local martingale and* $\langle \tilde{N} \rangle = \langle N \rangle$*, almost surely.*

(b) Conversely, given a continuous local martingale M*, define the exponential local martingale* Z *as above. If* Z *is a true martingale, then there exists a unique probability measure* $\mathbf{Q}$ *on* $(\Omega, \mathcal{F})$ *such that* $\mathbf{Q}^{(t)} = Z_t \cdot \mathbf{P}^{(t)}$ *for each* $t > 0$*.*

The final assertion in part (a) follows from the sum-of-squares approximation of the quadratic variation process $\langle N \rangle$; note that because $\mathbf{P} \sim \mathbf{Q}$, the notions "convergence in $\mathbf{Q}$ probability" and "convergence in $\mathbf{P}$ probability" are the same.

The case of constant H, say $H_s(\omega) = \mu$, is especially interesting. In this case Z is the familiar exponential martingale

$$Z_t = \exp(\mu B_t - \mu^2 t/2).$$

By the above discussion,

$$\beta_t := B_t - \mu t$$

is a $\mathbf{Q}$ local martingale. One of your final homework problems involves showing that $(\beta_t)_{t \ge 0}$ is in fact a Brownian motion under $\mathbf{Q}$. Granted this, we see that

$$B_t = \beta_t + \mu t,$$

so that the $\mathbf{Q}$ distribution of B is that of a Brownian motion (β_t) plus a *drift* μt.

More generally, let $f : \mathbf{R} \to \mathbf{R}$ be a bounded Borel measurable function. Then by results proved in class,

$$Z_t := \exp\left(\int_0^t f(B_s)\, dB_s - \frac{1}{2} \int_0^t [f(B_s)]^2\, ds \right)$$

is a positive martingale. Let $\mathbf{Q}$ be the associated probability measure:

$$\mathbf{Q}(A) = \int_A Z_t(\omega)\, \mathbf{P}(d\omega), \qquad A \in \mathcal{F}_t, t \ge 0.$$

By the preceding discussion and Levy's theorem, one can show that, under $\mathbf{Q}$,

$$\beta_t := B_t - \int_0^t f(B_s)\, ds$$

is standard Brownian motion. Therefore the canonical process B on the probability space $(\Omega, \mathcal{F}, \mathbf{Q})$ solves the "stochastic differential equation"

$$dX_t = d\beta_t + f(X_t)\, dt. \tag{2}$$