## Math 280C, Spring 2005

Foster-Liapunov Criterion

In what follows, $\mathbf{X} = (X_n)_{n=0}^{\infty}$ is a Markov chain with countable state space $S$ and transition probability matrix $P = \{p(x,y)\}_{x,y\in S}$. We suppose that $X$ has been constructed on the sequence space $\Omega = S^{\{0,1,2,\dots\}}$, and that $\mathbf{P}_x$ is the probability measure on $\Omega$ corresponding to the initial condition $X_0 = x$. Other notation is that used in class.

We present two criteria, both based on the following observation presented already in class.

**Proposition.** *Let $f : S \to [0,\infty)$ satisfy $Pf(x) \le f(x)$ for all $x \in S \setminus F$, where $F \subset S$. Define a stopping time by $D := \inf\{n \ge 0 : X_n \in F\}$. Then the stopped process $(f(X_{n\wedge D}))_{n\ge 0}$ is a $\mathbf{P}_x$-supermartingale for each $x \in S$.*

*Proof.* We compute

$$\begin{aligned}
\mathbf{E}_x[f(X_{(n+1)\wedge D})|\mathcal{F}_n] &= 1_{\{n<D\}}\mathbf{E}_x[f(X_{n+1})|\mathcal{F}_n] + \mathbf{E}_x[f(X_D)1_{\{D\le n\}}|\mathcal{F}_n] \\
&= 1_{\{n<D\}}\mathbf{E}_x[f(X_{n+1})|\mathcal{F}_n] + f(X_D)1_{\{D\le n\}} \\
&= 1_{\{n<D\}}Pf(X_n) + f(X_D)1_{\{D\le n\}} \\
&\le 1_{\{n<D\}}f(X_n) + f(X_D)1_{\{D\le n\}} \\
&= f(X_{n\wedge D}).
\end{aligned}$$

The inequality in the above computation follows from the hypothesis because $X_n \in S \setminus F$ when $n < D$. These conditional expectation calculations are valid even without knowing that $\mathbf{E}_x[f(X_{n\wedge D})]$ is finite, because $f \ge 0$. But having demonstrated the supermartingale inequality, we can now check the required integrability:

$$\mathbf{E}_x[f(X_{n\wedge D})] \le \mathbf{E}_x[f(X_{0\wedge D})] = f(x) < \infty.$$

∎

**Remark.** The same proof shows that $f(X_{n\wedge D})$ is a martingale provided $Pf = f$ on $S \setminus F$.

Here is a recurrence criterion for $\mathbf{X}$, based on the existence of a ("Liapunov") function on $S$ with certain properties. This type of criterion seems to have first appeared in the literature in a paper of F.G. Foster [**1**].

**Theorem 1.** *Assume that* $\mathbf{X}$ *is irreducible. Suppose there is a finite set* $F \subset S$ *and a function* $f : S \to [0, +\infty)$ *such that*

(a) $Pf(x) \le f(x)$ *for all* $x \notin F$, *and*

(b) $\{x \in S : f(x) \le M\}$ *is a finite set for each* $M > 0$.

*Then the Markov chain* $\mathbf{X}$ *is recurrent.*

*Proof.* Define stopping times $D := \inf\{n \ge 0 : X_n \in F\}$ and (for $M \in \mathbf{N}$) $S_M := \inf\{n \ge 0 : f(X_n) > M\}$.

Fix $x \in S$ and suppose that $\mathbf{P}_x[S_M = \infty] > 0$ for some $m \in \mathbf{N}$. Notice that

$$\{S_M = \infty\} \subset \{f(X_n) \le M \text{ for all } n\}.$$

Therefore, there is positive $\mathbf{P}_x$ probability that the Markov chain $\mathbf{X}$ remains in the *finite* set $\{x : f(x) \le M\}$ forever. This in turn implies that, with positive $\mathbf{P}_x$ probability, one of the states of $\{x : f(x) \le M\}$ is visited infinitely often. Such a state must be recurrent; we conclude that every element of $S$ is recurrent because $\mathbf{X}$ is irreducible. In short, if $\mathbf{P}_x[S_M = \infty] > 0$ for some $x \in S$, then $\mathbf{X}$ is recurrent and we are done.

Now suppose that $\mathbf{P}_x[S_M = \infty] = 0$ for all $x \in S$ and all $M \in \mathbf{N}$; that is, $\mathbf{P}_x[S_M < \infty] = 1$ for all $x \in S$ and all $M \in \mathbf{N}$. From class discussion we know that $f(X_{n\wedge D})$, $n \ge 0$, is a non-negative $\mathbf{P}_x$-supermartingale for all $x \in S$. By the optional stopping theorem for non-negative supermartingales we therefore have

$$\tag{1} \begin{aligned} f(x) = \mathbf{E}_x[f(X_{0\wedge D})] &\ge \mathbf{E}_x[f(X_{S_M \wedge D})] \\ &\ge \mathbf{E}_x[f(X_{S_M\wedge D}); S_M < D] \ge M \cdot \mathbf{P}_x[S_M < D], \end{aligned}$$

the final inequality following from the fact that $S_M \wedge D = S_M$ on $\{S_M < D\}$, and thus $f(X_{S_M\wedge D}) = f(X_{S_M}) \ge M$ on $\{S_M < D\}$. Comparing the extreme terms in (1) we arrive at

$$\tag{2} \mathbf{P}_x[S_M < D] \le f(x)/M, \qquad \forall x \in S, \forall M \in \mathbf{N}.$$

Observe that the events $\{S_M < D\}$ are nested inward; that is, $\{S_{M+1} < D\} \subset \{S_M < D\}$ for all $M \in \mathbf{N}$. Letting $M$ tend to $+\infty$ in (2) we therefore obtain

$$\tag{3} \mathbf{P}_x[\cap_{M\in\mathbf{N}}\{S_M < D\}] = 0, \qquad \forall x \in S.$$

Taking complements:

$$\tag{4} \mathbf{P}_x[D \le S_M \text{ for some } M \in \mathbf{N}] = 1, \qquad \forall x \in S.$$

When we combine (4) with the fact that $\mathbf{P}_x[S_M < \infty] = 1$ for all $x \in S$ and all $M \in \mathbf{N}$ we obtain

$$\text{(5)} \qquad \mathbf{P}_x[D < \infty] = 1, \qquad \forall x \in S.$$

Thus, return to $F$ is certain; by an argument used in class,

$$\text{(6)} \qquad \mathbf{P}_x[X_n \in F \text{ for infinitely many } n] = 1.$$

But $F$ is a finite set, so by the "pigeonhole principle", (6) implies that there exists $x_0 \in F$ such that

$$\mathbf{P}_x[X_n = x_0 \text{ for infinitely many } n] = 1.$$

Of course this state $x_0$ must be recurrent, and so $\mathbf{X}$ is recurrent because it is irreducible. $\square$

**Example 1.** Let $\{\xi_n\}_{n\geq 1}$ be an iid sequence of Bernoulli random variables: $\mathbf{P}[\xi_n = 1] = \mathbf{P}[\xi_n = -1] = 1/2$ for all $n$. Let $b : \mathbf{Z} \to \mathbf{Z}$ satisfy (i) $|b(x)| < |x|$ for all $x \neq 0$, (ii) $b(x) < 0$ for $x > 0$, and (iii) $b(x) > 0$ for $x < 0$. Consider the Markov chain $\mathbf{X} = \{X_n\}_{n\geq 0}$ generated recursively by

$$X_{n+1} = X_n + b(X_n) + \xi_{n+1}, \qquad n = 0, 1, 2, \ldots.$$

The hypotheses listed above ensure that the "drift" $b(x)$ tends to push $\mathbf{X}$ towards the state 0. Thus we expect $\mathbf{X}$ to be recurrent. Let us confirm this using Theorem 1. We take $f(x) := |x|$. Then

$$Pf(x) = \mathbf{E}_x|X_1| = \mathbf{E}_x|x + b(x) + \xi_1| = \frac{1}{2}\left(|x + b(x) + 1| + |x + b(x) - 1|\right).$$

Suppose that $x > 0$. Then by (i) and (ii) above we have $0 \leq -b(x) < x$, so $x + b(x) \pm 1 \geq 0$, whence

$$\frac{1}{2}\left(|x + b(x) + 1| + |x + b(x) - 1|\right) = x + b(x) < x = |x| = f(x),$$

which verifies that $Pf(x) \leq x$ if $x > 0$. In the same way $Pf(x) \leq f(x)$ if $x < 0$. Therefore Theorem 1 applies to this choice of $f$ with $F = \{0\}$. We conclude that $\mathbf{X}$ is recurrent. $\square$

The companion transience criterion is problem 5 on your second homework assignment:

**Theorem 2.** *Assume that* $\mathbf{X}$ *is irreducible. Suppose there is a finite set* $F$ *and a function* $g : S \to [0, +\infty)$ *such that*

*(a)* $Pg(x) \leq g(x)$ *for all* $x \notin F$*, and*

*(b)* $\inf\{g(x) : x \in S\} = 0$.

*Then* $\mathbf{X}$ *is transient.*

**Example 2.** Let us modify Example 1 by now assuming that $b(x) > 0$ if $x > 0$ and $b(x) < 0$ if $x < 0$. The drift $b(x)$ is now driving $\mathbf{X}$ away from 0, so we expect $\mathbf{X}$ to be transient. Use Theorem 2 to verify this intuition. □

An excellent source for results of the type presented here (and much more) is the book of Meyn and Tweedie [**2**].

## Reference


[1] F. G. Foster: On the stochastic matrices associated with certain queuing processes. *Ann. Math. Statistics* **24** (1953) 355–360.

[2] S.P. Meyn and R.L. Tweedie: *Markov Chains and Stochastic Stability*. Springer-Verlag, London, 1993.