Supplement on the Monotone Class Theorem

In dealing with integrals, the following “functional” form of the Monotone Class Theorem is often useful.

(1) **Theorem.** Let $\mathcal{K}$ be a collection of bounded real-valued functions on $\Omega$ that is closed under the formation of products (i.e., if $f, g \in \mathcal{K}$ then $fg \in \mathcal{K}$), and let $\mathcal{B}$ be the $\sigma$-algebra generated by $\mathcal{K}$. Let $\mathcal{H} \supset \mathcal{K}$ be a vector space (over $\mathbb{R}$) of bounded real-valued functions on $\Omega$ such that (a) $\mathcal{H}$ contains the constant functions and (b) if $(f_n) \subset \mathcal{H}$ with $\sup_{\omega} |f_n(\omega)| < +\infty$ and if $0 \leq f_1 \leq f_2 \leq \cdots \leq f_n \leq \cdots$, then $f := \lim_{n} f_n \in \mathcal{H}$. Under these conditions, $\mathcal{H}$ contains every bounded $\mathcal{B}$-measurable real-valued function on $\Omega$.

The proof of this theorem relies on the following

(2) **Lemma.** If $\mathcal{H}$ is as in the statement of Theorem (1), then $\mathcal{H}$ is closed under uniform convergence.

**Proof.** Suppose $(f_n) \subset \mathcal{H}$ and $f_n \to f$ uniformly on $\Omega$. (That is, $\lim_n \sup_{\omega} |f_n(\omega) - f(\omega)| = 0$.) By passing to a subsequence if necessary, we can arrange that $\|f_{n+1} - f_n\|_{\infty} \leq 2^{-n}$; the subsequence still converges to $f$. Define $g_n := f_n - 2^{1-n} + 2$. Then $g_n \in \mathcal{H}$ since $\mathcal{H}$ is a vector space containing the constant functions, and

$$\sup_{\omega} |g_n(\omega)| \leq \sup_{\omega} |f_n(\omega)| + 2,$$

so the sequence $(g_n)$ is uniformly bounded. Also, $g_{n+1} - g_n = f_{n+1} - f_n + 2^{-n} \geq 0$ ($n = 1, 2, \ldots$) and $g_1 = f_1 + 1 \geq 0$. It follows that $(g_n)$ is a (uniformly bounded) increasing sequence, with $\lim_n g_n = \lim_n f_n + 2 = f + 2$. But $\lim_n g_n \in \mathcal{H}$, hence so is $f = \lim_n g_n - 2$. $\square$

**Proof of (1).** Owing to the closure properties of $\mathcal{H}$ and the fact that every non-negative $\mathcal{B}$-measurable function is the pointwise limit of an increasing sequence of simple functions, it suffices to show that $\mathcal{H}$ contains the indicator $1_D$ of every $D \in \mathcal{B}$. Define $\mathcal{L} := \{D \in \mathcal{B} : 1_D \in \mathcal{H}\}$. It is easy to see that $\mathcal{L}$ is a $\lambda$-system. We are going to show that $\mathcal{L}$ contains a $\pi$-system $\mathcal{P}$ generating $\mathcal{B}$. In view of the Monotone Class Theorem, this will imply $\mathcal{L} \supset \mathcal{B}$, whence $\mathcal{L} = \mathcal{B}$.

Let $\mathcal{A}_0$ denote the algebra* generated by $\mathcal{K}$; since $\mathcal{K}$ is already closed under products, $\mathcal{A}_0$ is simply the linear span of $\mathcal{K}$. Consequently, $\mathcal{A}_0 \subset \mathcal{H}$. By Lemma (2), the uniform closure $\mathcal{A}$ of $\mathcal{A}_0$ is also contained in $\mathcal{H}$. Referring to the standard proof of Weierstrass’ Theorem, we see that if $f \in \mathcal{A}$, then $|f| \in \mathcal{A}$ as well. Consequently, if $f, g \in \mathcal{A}$, then $f \vee g = (|f - g| + f + g)/2$ and $f \wedge g = (f + g - |f - g|)/2$ are elements of $\mathcal{A}$. Now fix $f \in \mathcal{A}$ and $b \in \mathbb{R}$. Then for each $n \in \mathbb{N}$ the function $\varphi_n := [n(f - b)^+] \wedge 1$ is an element of $\mathcal{A}$, hence an element of $\mathcal{H}$. As $n \to \infty$, $\varphi_n$ increases pointwise to $1_{\{f > b\}}$. Thus, since $\mathcal{H}$ is closed under bounded monotone convergence, $1_{\{f > b\}} \in \mathcal{H}$; this means that $\{f > b\} \in \mathcal{L}$. More generally, if $\{f_1, f_2, \ldots, f_m\}$ is a finite sequence of functions from $\mathcal{A}$ and if $\{b_1, b_2, \ldots, b_m\}$ is a finite sequence of real numbers, then the function

$$g_n := \prod_{k=1}^{m} [n(f_k - b_k)^+] \wedge 1$$

* An algebra (of functions) is a vector space that is also closed under the formation of products.
is an element of $A$, and the sequence $\{g_n\}$ increases boundedly and pointwise to $\prod_{k=1}^m 1_{\{f_k > b_k\}}$, which is therefore an element of $H$, as before. This function is the indicator of the set $B := \cap_{k=1}^m \{\omega : f_k(\omega) > b_k\}$, so $B \in \mathcal{L}$. It follows that $\mathcal{L}$ contains the $\pi$-system $\mathcal{P}$ consisting of finite intersections of sets of the form $f^{-1}(I)$, where $f \in A$ and $I \subset \mathbb{R}$ is an open right-halfline. Since $\mathcal{K} \subset A$, the $\sigma$-algebra generated by $\mathcal{P}$ is $B (= \sigma(\mathcal{K}))$. We have constructed a $\pi$-system generating $B$ and contained in $\mathcal{L}$, as desired. \end{proof}

(3) Exercise. Let $(\Omega, \mathcal{B})$ be a measurable space. Let $\mathcal{K}$ be a collection of bounded $\mathcal{B}$-measurable real-valued functions on $\Omega$ such that $\mathcal{B}$ is the $\sigma$-algebra generated by $\mathcal{K}$. Assume that $\mathcal{K}$ is closed under the formation of products. Let $P$ and $Q$ be two probability measures on $(\Omega, \mathcal{B})$ such that $\int X dP = \int X dQ$ for all $X \in \mathcal{K}$. Prove that $\int X dP = \int X dQ$ for every bounded $\mathcal{B}$-measurable random variable $X$. In particular, $P = Q$ on $\mathcal{B}$. [Hint: Take $\mathcal{H}$ to be the class of bounded $\mathcal{B}$-measurable random variables $X$ such that $\int X dP = \int X dQ$, and apply Theorem (1).]

(4) Example. Let $(X, \mathcal{M})$ and $(Y, \mathcal{N})$ be measurable spaces, and recall that $\mathcal{M} \otimes \mathcal{N}$ denotes the product $\sigma$-algebra on the cartesian product $X \times Y$. More precisely, $\mathcal{M} \otimes \mathcal{N}$ is the $\sigma$-algebra on $X \times Y$ generated by the projections $\pi_1, \pi_2$, where $\pi_1(x, y) = x$ and $\pi_2(x, y) = y$ for $(x, y) \in X \times Y$. Let $\mathcal{K}$ denote the set of functions of the form $(x, y) \mapsto f(x)g(y)$, where $f$ (resp. $g$) is a bounded real-valued $\mathcal{M}$-measurable (resp. $\mathcal{N}$-measurable) function on $X$ (resp. $Y$). Clearly the $\sigma$-algebra generated by $\mathcal{K}$ is just $\mathcal{M} \otimes \mathcal{N}$. Now let $\mathcal{H}$ be the set of bounded real-valued $\mathcal{M} \otimes \mathcal{N}$-measurable functions $h$ such that $x \mapsto h(x, y)$ is $\mathcal{M}$-measurable for each fixed $y \in Y$. It is a simple matter to check that $\mathcal{H}$ is a vector space satisfying the conditions of Theorem (1). As a consequence of that result, $\mathcal{H}$ is precisely the class of all bounded real-valued $\mathcal{M} \otimes \mathcal{N}$-measurable functions. This is an alternative proof of one of the measurability assertions in Tonelli’s theorem, at least for bounded functions. A truncation argument reduces the general case to the bounded case.

(5) Exercise. Let $(\Omega, \mathcal{B}, P)$ be a probability space, and let $X$ and $Y$ be random variables defined on $(\Omega, \mathcal{B})$. Suppose that

$E[f(X)g(Y)] = E[f(X)g(X)]$

for every pair $(f, g)$ of bounded continuous functions from $\mathbb{R}$ to $\mathbb{R}$. Prove that $P[X = Y] = 1$. [Hint: Use Theorem (1) to show that $E[h(X, Y)] = E[h(X, X)]$ for every bounded $\mathcal{B}(\mathbb{R}^2)$-measurable real-valued function $h$. Then consider $h = 1_{\Delta}$, where $\Delta = \{(x, x) : x \in \mathbb{R}\}$ is the “diagonal” in $\mathbb{R}^2$.]

The following variant of Theorem (1) is sometimes useful. The proof is quite similar to that of Theorem (1), and so is omitted.

(7) Theorem. Let $\mathcal{C}$ be an algebra of bounded real-valued functions on $\Omega$ that contains the constant functions, and let $\mathcal{B}$ be the $\sigma$-algebra generated by $\mathcal{C}$. Let $\mathcal{H} \supset \mathcal{C}$ be a set of bounded real-valued functions on $X$ that is closed under bounded monotone convergence and uniform convergence. Under these conditions, $\mathcal{H}$ contains every bounded $\mathcal{B}$-measurable real-valued function on $\Omega$.

As an application of Theorem (7), you are asked to prove a functional form of Exercise 5 from Chapter 2.
(8) Exercise. Let \((S,d)\) be a metric space, let \(BC(S)\) denote the class of bounded continuous real-valued functions on \(S\), and let \(B(S)\) denote the Borel \(\sigma\)-field on \(S\); thus \(B(S)\) is the \(\sigma\)-field generated by the open subsets of \(S\).

(a) Verify that \(B(S)\) is the \(\sigma\)-algebra generated by \(BC(S)\).

(b) Let \(\mu\) be a probability measure on \((S,B(S))\) and let \(f\) be a bounded real-valued \(B(S)\)-measurable function. Using Theorem (7), show that for each \(\epsilon > 0\) there exists \(g \in BC(S)\) such that \(\int |f - g| \, d\mu \leq \epsilon\). [Hint: Take \(\mathcal{H}\) to be the class of bounded real-valued \(B(S)\)-measurable functions for which the asserted approximation property holds, and take \(C = BC(S)\). Show that \(\mathcal{H}\) has the required closure properties for Theorem (7) to apply.]

(9) Example. Let \(\{X_t : t \in T\}\) be a collection of random variables, indexed by \(T\), defined on a common measurable space \((\Omega, \mathcal{B})\). Let \(\mathcal{X} \subset \mathcal{B}\) denote the \(\sigma\)-field generated by \(\{X_t : t \in T\}\).

Claim: If \(F : \Omega \to \mathbb{R}\) is \(\mathcal{X}\)-measurable, then there is a sequence \(\{t_1, t_2, \ldots\} \subset T\) and a \(B(\mathbb{R}^N)\)-measurable function \(f : \mathbb{R}^N \to \mathbb{R}\) such that

\[
F(\omega) = f(X_{t_1}(\omega), X_{t_2}(\omega), \ldots), \quad \forall \omega \in \Omega.
\]

To see the Claim apply Theorem (1) with \(\mathcal{K}\) equal to the class of functions of the form

\[
\omega \mapsto \prod_{k=1}^n f_k(X_{t_k}(\omega)),
\]

where \(n \in \mathbb{N}\), \(t_k \in T\), and each \(f_k : \mathbb{R} \to \mathbb{R}\) is bounded and \(B(\mathbb{R})\)-measurable. Observe that \(\sigma(\mathcal{K}) = \mathcal{X}\). Take \(\mathcal{H}\) to be the class of bounded \(\mathcal{X}\)-measurable functions for which a representation like (10) holds. It should be clear that \(\mathcal{H}\) is a vector space containing the constant functions. Let \(\{F_n\}_{n \in \mathbb{N}}\) be a uniformly bounded increasing sequence of elements of \(\mathcal{H}\). Since a countable union of countable sets is itself countable, we may suppose that there is a single sequence \(\{t_1, t_2, \ldots\}\) such that

\[
F_n(\omega) = f_n(X_{t_1}(\omega), X_{t_2}(\omega), \ldots), \quad \forall n \in \mathbb{N}, \omega \in \Omega,
\]

where each \(f_n\) is bounded and \(B(\mathbb{R}^N)\)-measurable. Since \(\{F_n\}\) is uniformly bounded, we can suppose that there is a constant \(M\) such that \(|f_n(x)| \leq M\) for all \(n \in \mathbb{N}\) and all \(x \in \mathbb{R}^N\). Define \(f : \mathbb{R}^N \to \mathbb{R}\) by

\[
f(x) := \lim \inf_{n \to \infty} f_n(x), \quad x \in \mathbb{R}^N.
\]

Then \(f\) is bounded in magnitude by \(M\) and is \(B(\mathbb{R}^N)\)-measurable. Moreover, because \(F_n(\omega)\) increases with \(n\),

\[
F(\omega) := \lim_n F_n(\omega) = \lim \inf_n F_n(\omega) = \lim \inf_n f_n(X_{t_1}(\omega), X_{t_2}(\omega), \ldots) = f(X_{t_1}(\omega), X_{t_2}(\omega), \ldots),
\]

so \(F\) admits a representation as in (10). That is, \(\mathcal{H}\) is closed under bounded monotone convergence.

By Theorem (1), \(\mathcal{H}\) contains every bounded \(\mathcal{X}\)-measurable real-valued function on \(\Omega\). This proves the Claim.

Another variant of Theorem 1 concerns non-negative functions. A collection \(\mathcal{D}\) of functions on \(\Omega\) is a convex cone provided: If \(f, g \in \mathcal{D}\) and \(\alpha, \beta \geq 0\) then \(\alpha f + \beta g \in \mathcal{D}\). The original result of this type may be due to E.B. Dynkin, but the version stated here is taken from the monograph *A User’s Guide to Measure Theoretic Probability* by D. Pollard.
Theorem. Let $K^+$ be a collection of bounded non-negative real-valued functions on $\Omega$ that is closed under the formation of products, and let $\mathcal{B}$ be the $\sigma$-algebra generated by $K^+$. Let $\mathcal{H}^+ \supseteq K^+$ be a convex cone of bounded non-negative real-valued functions on $\Omega$ such that (a) $\mathcal{H}^+$ contains the non-negative constant functions, (b) if $(f_n) \subseteq \mathcal{H}^+$ with $\sup_n \sup_\omega |f_n(\omega)| < +\infty$ and if $0 \leq f_1 \leq f_2 \leq \cdots \leq f_n \leq \cdots$, then $f := \lim_n f_n \in \mathcal{H}^+$, and (c) if $f, g \in \mathcal{H}^+$ and $f \geq g$ then $f - g \in \mathcal{H}^+$. Under these conditions, $\mathcal{H}^+$ contains every bounded, non-negative, $\mathcal{B}$-measurable real-valued function on $\Omega$.

The proof is a minor modification of the proof of Theorem 1. Let $C_0$ be the convex cone generated by $K^+$ and let $C \subset \mathcal{H}^+$ be its uniform closure. Both $C_0$ and $C$ are closed under the formation of products.

Claim. If $f \in C$ and $\varphi : [0, \infty) \to [0, \infty)$ is continuous, then $\varphi f \in \mathcal{H}^+$.

To see this let $M := \sup \{f(x) : x \in \Omega\}$, and using the Weierstrass approximation theorem choose a sequence of real polynomials $(p_n)$ such that $\sup_{x \in [0,2M]} |p_n(x) - \varphi(x)| \leq n^{-1}$. Owing to the closure properties of $C$ and $\mathcal{H}^+$, $x \mapsto n^{-1} + p_n \varphi \in \mathcal{H}^+$ for each $n$, and so the uniform limit $\varphi f = \lim_n n^{-1} + p_n \varphi \in \mathcal{H}^+$ as well; cf. Lemma 2. By using $t \mapsto \lfloor n(t - b) \rfloor \wedge 1$ ($n = 1, 2, \ldots, b \geq 0$) as in the proof of Theorem 1, we see that if $f \in C$ then $1_{\{f > b\}} \in \mathcal{H}^+$ for each $b \geq 0$, and the rest of the proof goes as before.

Exercise. Let $(\Omega, \mathcal{B})$ be a measurable space. Let $K^+$ be a collection of bounded non-negative $\mathcal{B}$-measurable real-valued functions on $\Omega$ such that $\mathcal{B}$ is the $\sigma$-algebra generated by $K^+$. Assume that $K^+$ is closed under the formation of products. Let $\mu$ and $\nu$ be two measures on $(\Omega, \mathcal{B})$ such that $\int X \, d\mu = \int X \, d\nu < \infty$ for all $X \in K^+$. Prove that $\int X \, dP = \int X \, dQ$ for every bounded non-negative $\mathcal{B}$-measurable function $X$. In particular, $P = Q$ on $\mathcal{B}$. [Hint: Take $\mathcal{H}^+$ to be the class of bounded non-negative $\mathcal{B}$-measurable functions $X$ such that $\int X \, dP = \int X \, dQ$, and apply Theorem (11).]