Math 280A **Fall 2021**

Supplement on the Monotone Class Theorem

In dealing with integrals, the following "functional" form of the Monotone Class Theorem is often useful.

(1) Theorem. *Let $\mathcal{K}$ be a collection of bounded real-valued functions on Ω that is closed under the formation of products (i.e., if $f, g \in \mathcal{K}$ then $fg \in \mathcal{K}$), and let $\mathcal{B}$ be the σ-algebra generated by $\mathcal{K}$. Let $\mathcal{H} \supset \mathcal{K}$ be a vector space (over $\mathbf{R}$) of bounded real-valued functions on Ω such that (a) $\mathcal{H}$ contains the constant functions and (b) if $(f_n) \subset \mathcal{H}$ with $\sup_n \sup_\omega |f_n(\omega)| < +\infty$ and if $0 \le f_1 \le f_2 \le \cdots \le f_n \le \cdots$, then $f := \lim_n f_n \in \mathcal{H}$. Under these conditions, $\mathcal{H}$ contains every bounded $\mathcal{B}$-measurable real-valued function on Ω.*

The proof of this theorem relies on the following

(2) Lemma. *If $\mathcal{H}$ is as in the statement of Theorem (1), then $\mathcal{H}$ is closed under uniform convergence.*

Proof. Suppose $(f_n) \subset \mathcal{H}$ and $f_n \to f$ uniformly on Ω. (That is, $\lim_n \sup_\omega |f_n(\omega) - f(\omega)| = 0$.) By passing to a subsequence if necessary, we can arrange that $\|f_{n+1} - f_n\|_\infty \le 2^{-n}$; the subsequence still converges to f. Define $g_n := f_n - 2^{1-n} + 2$. Then $g_n \in \mathcal{H}$ since $\mathcal{H}$ is a vector space containing the constant functions, and

$$\sup_\omega |g_n(\omega)| \le \sup_\omega |f_n(\omega)| + 2,$$

so the sequence (g_n) is uniformly bounded. Also, $g_{n+1} - g_n = f_{n+1} - f_n + 2^{-n} \ge 0$ $(n = 1, 2, \ldots)$ and $g_1 = f_1 + 1 \ge 0$. It follows that (g_n) is a (uniformly bounded) increasing sequence, with $\lim_n g_n = \lim_n f_n + 2 = f + 2$. But $\lim_n g_n \in \mathcal{H}$, hence so is $f = \lim_n g_n - 2$. □

Proof of (1). Owing to the closure properties of $\mathcal{H}$ and the fact that every non-negative $\mathcal{B}$-measurable function is the pointwise limit of an increasing sequence of simple functions, it suffices to show that $\mathcal{H}$ contains the indicator 1_D of every $D \in \mathcal{B}$. Define $\mathcal{L} := \{D \in \mathcal{B} : 1_D \in \mathcal{H}\}$. It is easy to see that $\mathcal{L}$ is a λ-system. We are going to show that $\mathcal{L}$ contains a π-system $\mathcal{P}$ generating $\mathcal{B}$. In view of the Monotone Class Theorem, this will imply $\mathcal{L} \supset \mathcal{B}$, whence $\mathcal{L} = \mathcal{B}$.

Let $\mathcal{A}_0$ denote the algebra* generated by $\mathcal{K}$; since $\mathcal{K}$ is already closed under products, $\mathcal{A}_0$ is simply the linear span of $\mathcal{K}$. Consequently, $\mathcal{A}_0 \subset \mathcal{H}$. By Lemma (2), the uniform closure $\mathcal{A}$ of $\mathcal{A}_0$ is also contained in $\mathcal{H}$. Referring to the standard proof of Weierstrass' Theorem, we see that if $f \in \mathcal{A}$, then $|f| \in \mathcal{A}$ as well. Consequently, if $f, g \in \mathcal{A}$, then $f \vee g = [|f - g| + f + g]/2$ and $f \wedge g = [f + g - |f - g|]/2$ are elements of $\mathcal{A}$. Now fix $f \in \mathcal{A}$ and $b \in \mathbf{R}$. Then for each $n \in \mathbf{N}$ the function $\varphi_n := [n(f - b)^+] \wedge 1$ is an element of $\mathcal{A}$, hence an element of $\mathcal{H}$. As $n \to \infty$, φ_n increases pointwise to $1_{\{f > b\}}$. Thus, since $\mathcal{H}$ is closed under bounded monotone convergence, $1_{\{f>b\}} \in \mathcal{H}$; this means that $\{f > b\} \in \mathcal{L}$. More generally, if $\{f_1, f_2, \ldots, f_m\}$ is a finite sequence of functions from $\mathcal{A}$ and if $\{b_1, b_2, \ldots, b_m\}$ is a finite sequence of real numbers, then the function

$$g_n := \prod_{k=1}^{m} [n(f_k - b_k)^+] \wedge 1$$

* An *algebra* (of functions) is a vector space that is also closed under the formation of products.

is an element of $\mathcal{A}$, and the sequence $\{g_n\}$ increases boundedly and pointwise to $\prod_{k=1}^m 1_{\{f_k > b_k\}}$, which is therefore an element of $\mathcal{H}$, as before. This function is the indicator of the set $B := \cap_{k=1}^m \{\omega : f_k(\omega) > b_k\}$, so $B \in \mathcal{L}$. It follows that $\mathcal{L}$ contains the π-system $\mathcal{P}$ consisting of finite intersections of sets of the form $f^{-1}(I)$, where $f \in \mathcal{A}$ and $I \subset \mathbf{R}$ is an open right-halfline. Since $\mathcal{K} \subset \mathcal{A}$, the σ-algebra generated by $\mathcal{P}$ is $\mathcal{B}$ $(= \sigma(\mathcal{K}))$. We have constructed a π-system generating $\mathcal{B}$ and contained in $\mathcal{L}$, as desired. □

(3) Exercise. Let $(\Omega, \mathcal{B})$ be a measurable space. Let $\mathcal{K}$ be a collection of bounded $\mathcal{B}$-measurable real-valued functions on Ω such that $\mathcal{B}$ is the σ-algebra generated by $\mathcal{K}$. Assume that $\mathcal{K}$ is closed under the formation of products. Let P and Q be two probability measures on $(\Omega, \mathcal{B})$ such that $\int X\,dP = \int X\,dQ$ for all $X \in \mathcal{K}$. Prove that $\int X\,dP = \int X\,dQ$ for every bounded $\mathcal{B}$-measurable random variable X. In particular, $P = Q$ on $\mathcal{B}$. [Hint: Take $\mathcal{H}$ to be the class of bounded $\mathcal{B}$-measurable random variables X such that $\int X\,dP = \int X\,dQ$, and apply Theorem (1).]

(4) Example. Let $(X, \mathcal{M})$ and $(Y, \mathcal{N})$ be measurable spaces, and recall that $\mathcal{M} \otimes \mathcal{N}$ denotes the product σ-algebra on the cartesian product $X \times Y$. More precisely, $\mathcal{M} \otimes \mathcal{N}$ is the σ-algebra on $X \times Y$ generated by the projections π_1, π_2, where $\pi_1(x, y) = x$ and $\pi_2(x, y) = y$ for $(x, y) \in X \times Y$. Let $\mathcal{K}$ denote the set of functions of the form $(x, y) \mapsto f(x)g(y)$, where f (resp. g) is a bounded real-valued $\mathcal{M}$-measurable (resp. $\mathcal{N}$-measurable) function on X (resp. Y). Clearly the σ-algebra generated by $\mathcal{K}$ is just $\mathcal{M} \otimes \mathcal{N}$. Now let $\mathcal{H}$ be the set of bounded real-valued $\mathcal{M} \otimes \mathcal{N}$-measurable functions h such that $x \mapsto h(x, y)$ is $\mathcal{M}$-measurable for each fixed $y \in Y$. It is a simple matter to check that $\mathcal{H}$ is a vector space satisfying the conditions of Theorem (1). As a consequence of that result, $\mathcal{H}$ is precisely the class of *all* bounded real-valued $\mathcal{M} \otimes \mathcal{N}$-measurable functions. This is an alternative proof of one of the measurability assertions in Tonelli's theorem, at least for bounded functions. A truncation argument reduces the general case to the bounded case.

(5) Exercise. Let $(\Omega, \mathcal{B}, P)$ be a probability space, and let X and Y be random variables defined on $(\Omega, \mathcal{B})$. Suppose that

$$E[f(X)g(Y)] = E[f(X)g(X)] \tag{6}$$

for every pair (f, g) of bounded continuous functions from $\mathbf{R}$ to $\mathbf{R}$. Prove that $P[X = Y] = 1$. [Hint: Use Theorem (1) to show that $E[h(X, Y)] = E[h(X, X)]$ for every bounded $\mathcal{B}(\mathbf{R}^2)$-measurable real-valued function h. Then consider $h = 1_\Delta$, where $\Delta = \{(x, x) : x \in \mathbf{R}\}$ is the "diagonal" in $\mathbf{R}^2$.]

The following variant of Theorem (1) is sometimes useful. The proof is quite similar to that of Theorem (1), and so is omitted.

(7) Theorem. *Let $\mathcal{C}$ be an algebra of bounded real-valued functions on Ω that contains the constant functions, and let $\mathcal{B}$ be the σ-algebra generated by $\mathcal{C}$. Let $\mathcal{H} \supset \mathcal{C}$ be a set of bounded real-valued functions on X that is closed under bounded monotone convergence and uniform convergence. Under these conditions, $\mathcal{H}$ contains every bounded $\mathcal{B}$-measurable real-valued function on Ω.*

As an application of Theorem (7), you are asked to prove a functional form of Exercise 5 from Chapter 2.

(8) Exercise. Let (S, d) be a metric space, let $BC(S)$ denote the class of bounded continuous real-valued functions on S, and let $\mathcal{B}(S)$ denote the Borel σ-field on S; thus $\mathcal{B}(S)$ is the σ-field generated by the open subsets of S.

(a) Verify that $\mathcal{B}(S)$ is the σ-algebra generated by $BC(S)$.

(b) Let μ be a probability measure on $(S, \mathcal{B}(S))$ and let f be a bounded real-valued $\mathcal{B}(S)$-measurable function. Using Theorem (7), show that for each $\epsilon > 0$ there exists $g \in BC(S)$ such that $\int |f - g| \, d\mu \le \epsilon$. [Hint: Take $\mathcal{H}$ to be the class of bounded real-valued $\mathcal{B}(S)$-measurable functions for which the asserted approximation property holds, and take $\mathcal{C} = BC(S)$. Show that $\mathcal{H}$ has the required closure properties for Theorem (7) to apply.]

(9) Example. Let $\{X_t : t \in \mathbf{T}\}$ be a collection of random variables, indexed by $\mathbf{T}$, defined on a common measurable space $(\Omega, \mathcal{B})$. Let $\mathcal{X} \subset \mathcal{B}$ denote the σ-field generated by $\{X_t : t \in \mathbf{T}\}$.

Claim: If $F : \Omega \to \mathbf{R}$ is $\mathcal{X}$-measurable, then there is a sequence $\{t_1, t_2, \ldots\} \subset \mathbf{T}$ and a $\mathcal{B}(\mathbf{R}^{\mathbf{N}})$-measurable function $f : \mathbf{R}^{\mathbf{N}} \to \mathbf{R}$ such that

$$F(\omega) = f(X_{t_1}(\omega), X_{t_2}(\omega), \ldots), \qquad \forall \omega \in \Omega. \tag{10}$$

To see the Claim apply Theorem (1) with $\mathcal{K}$ equal to the class of functions of the form

$$\omega \mapsto \prod_{k=1}^{n} f_k(X_{t_k}(\omega))),$$

where $n \in \mathbf{N}$, $t_k \in \mathbf{T}$, and each $f_k : \mathbf{R} \to \mathbf{R}$ is bounded and $\mathcal{B}(\mathbf{R})$-measurable. Observe that $\sigma(\mathcal{K}) = \mathcal{X}$. Take $\mathcal{H}$ to be the class of bounded $\mathcal{X}$-measurable functions for which a representation like (10) holds. It should be clear that $\mathcal{H}$ is a vector space containing the constant functions. Let $\{F_n\}_{n \in \mathbf{N}}$ be a uniformly bounded increasing sequence of elements of $\mathcal{H}$. Since a countable union of countable sets is itself countable, we may suppose that there is a single sequence $\{t_1, t_2, \ldots\}$ such that

$$F_n(\omega) = f_n(X_{t_1}(\omega), X_{t_2}(\omega), \ldots), \qquad \forall n \in \mathbf{N}, \omega \in \Omega,$$

where each f_n is bounded and $\mathcal{B}(\mathbf{R}^{\mathbf{N}})$-measurable. Since $\{F_n\}$ is uniformly bounded, we can suppose that there is a constant M such that $|f_n(\mathbf{x})| \le M$ for all $n \in \mathbf{N}$ and all $\mathbf{x} \in \mathbf{R}^{\mathbf{N}}$. Define $f : \mathbf{R}^{\mathbf{N}} \to \mathbf{R}$ by

$$f(\mathbf{x}) := \liminf_{n \to \infty} f_n(\mathbf{x}), \qquad \mathbf{x} \in \mathbf{R}^{\mathbf{N}}.$$

Then f is bounded in magnitude by M and is $\mathcal{B}(\mathbf{R}^{\mathbf{N}})$-measurable. Moreover, because $F_n(\omega)$ increases with n,

$$F(\omega) := \lim_n F_n(\omega) = \liminf_n F_n(\omega) = \liminf_n f_n(X_{t_1}(\omega), X_{t_2}(\omega), \ldots) = f(X_{t_1}(\omega), X_{t_2}(\omega), \ldots),$$

so F admits a representation as in (10). That is, $\mathcal{H}$ is closed under bounded monotone convergence. By Theorem (1), $\mathcal{H}$ contains every bounded $\mathcal{X}$-measurable real-valued function on Ω. This proves the Claim.

Another variant of Theorem 1 concerns non-negative functions. A collection $\mathcal{D}$ of functions on Ω is a *convex cone* provided: If $f, g \in \mathcal{D}$ and $\alpha, \beta \ge 0$ then $\alpha f + \beta g \in \mathcal{D}$. The original result of this type may be due to E.B. Dynkin, but the version stated here is taken from the monograph *A User's Guide to Measure Theoretic Probability* by D. Pollard.

(11) Theorem. *Let $\mathcal{K}^+$ be a collection of bounded non-negative real-valued functions on Ω that is closed under the formation of products, and let $\mathcal{B}$ be the σ-algebra generated by $\mathcal{K}^+$. Let $\mathcal{H}^+ \supset \mathcal{K}^+$ be a convex cone of bounded non-negative real-valued functions on Ω such that* (a) *$\mathcal{H}^+$ contains the non-negative constant functions,* (b) *if $(f_n) \subset \mathcal{H}^+$ with $\sup_n \sup_\omega |f_n(\omega)| < +\infty$ and if $0 \le f_1 \le f_2 \le \cdots \le f_n \le \cdots$, then $f := \lim_n f_n \in \mathcal{H}^+$, and* (c) *if $f, g \in \mathcal{H}^+$ and $f \ge g$ then $f - g \in \mathcal{H}^+$. Under these conditions, $\mathcal{H}^+$ contains every bounded, non-negative, $\mathcal{B}$-measurable real-valued function on Ω.*

The proof is a minor modification of the proof of Theorem 1. Let $\mathcal{C}_0$ be the convex cone generated by $\mathcal{K}^+$ and let $\mathcal{C} \subset \mathcal{H}^+$ be its uniform closure. Both $\mathcal{C}_0$ and $\mathcal{C}$ are closed under the formation of. products.

Claim. If $f \in \mathcal{C}$ and $\varphi : [0, \infty) \to [0, \infty)$ is continuous, then $\varphi \circ f \in \mathcal{H}^+$.

To see this let $M := \sup\{f(x) : x \in \Omega\}$, and using the Weierstrass approximation theorem choose a sequence of real polynomials (p_n) such that $\sup_{x \in [0,2M]} |p_n(x) - \varphi(x)| \le n^{-1}$. Owing to the closure properties of $\mathcal{C}$ and $\mathcal{H}^+$, $x \mapsto n^{-1} + p_n \circ f \in \mathcal{H}^+$ for each n, and so the uniform limit $\varphi \circ f = \lim_n n^{-1} + p_n \circ f \in \mathcal{H}^+$ as well; *cf.* Lemma 2. By using $t \mapsto [n(t-b)^+] \wedge 1$ ($n = 1, 2, \ldots, b \ge 0$) as in the proof of Theorem 1, we see that if $f \in \mathcal{C}$ then $1_{\{f > b\}} \in \mathcal{H}^+$ for each $b \ge 0$, and the rest of the proof goes as before.

(12) Exercise. Let $(\Omega, \mathcal{B})$ be a measurable space. Let $\mathcal{K}^+$ be a collection of bounded non-negative $\mathcal{B}$-measurable real-valued functions on Ω such that $\mathcal{B}$ is the σ-algebra generated by $\mathcal{K}^+$. Assume that $\mathcal{K}^+$ is closed under the formation of products. Let μ and ν be two measures on $(\Omega, \mathcal{B})$ such that $\int X \, d\mu = \int X \, d\nu < \infty$ for all $X \in \mathcal{K}^+$. Prove that $\int X \, dP = \int X \, dQ$ for every bounded non-negative $\mathcal{B}$-measurable function X. In particular, $P = Q$ on $\mathcal{B}$. [Hint: Take $\mathcal{H}^+$ to be the class of bounded non-negative $\mathcal{B}$-measurable functions X such that $\int X \, dP = \int X \, dQ$, and apply Theorem (11).]