Remark on a paper of Adamou and Peters

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This note concerns a cooperative growth model discussed in a recent paper [**1**] of Adamou & Peters.

Consider the discrete time stochastic process (X_n) define by $X_0 = 1$ and

$$X_n := \prod_{k=1}^{n} \xi_k, \qquad n = 1, 2, \ldots,$$

where the ξ_k are iid, strictly positive, non-degenerate, with finite mean μ. Clearly

$$\mathbf{E}[X_n] = \mu^n,$$

while by the Strong Law of Large Numbers

$$\lim_n \frac{1}{n} \log X_n = \ell, \qquad \text{almost surely,}$$

where

$$\ell := \mathbf{E}[\log \xi_k] < \log \mu,$$

the inequality owing to Jensen. That is, almost surely,

$$X_n = e^{n\ell + o(n)},$$

which grows more slowly than its average μ^n.

Now consider a second independent identically distributed growth process

$$Y_n = \prod_{k=1}^{n} \eta_k, \qquad n = 1, 2, \ldots.$$

Of course (Y_n) will evolve as (X_n), statistically. But suppose these two growth processes "cooperate" by pooling resources after each time step. That is, consider a third process (Z_n) defined recursively by $Z_0 = 1$ and

$$Z_{n+1} = Z_n \cdot \frac{\xi_{n+1} + \eta_{n+1}}{2}.$$

Evidently,

$$Z_n = \prod_{k=1}^{n} \theta_k,$$

where $\theta_k := (\xi_k + \eta_k)/2$. Of course, $\mathbf{E}[\theta_k] = \mu$, but

$$\tilde{\ell} := \mathbf{E}[\log \theta_k] = \mathbf{E}\left[\log\left(\frac{\xi_k + \eta_k}{2}\right)\right] > \mathbf{E}\left[\frac{\log \xi_k + \log \eta_k}{2}\right] = \ell,$$

the inequality again a consequence of Jensen. Thus, although $\mathbf{E}[Z_n] = \mathbf{E}[X_n] = \mathbf{E}[Y_n]$ for all n, we have that

$$\frac{Z_n}{X_n} = e^{n(\tilde{\ell}-\ell)+o(n)}$$

almost surely. In short, pooling (which reduces variability) enhances growth significantly. For example, if $\mathbf{P}[\xi_k = 0.82] = \mathbf{P}[\xi_k = 1.20] = 1/2$, then $\mu = 1.01$, $\ell = -.00806...$, and $\tilde{\ell} = 0.000942...$, so that cooperation may turn long-term decay into long-term growth!

Needless to say, the same analysis leads to the same conclusion for more than two cooperators.

The above is the main result of [**1**], which contains further discussion and extensions. The authors of [**1**] confine their attention to the situation in which ξ_k has the log-normal distribution, so that ℓ and $\tilde{\ell}$ can be computed explicitly—Jensen's inequality plays no role.

References

[1] O. Peters, A. Adamou: An evolutionary advantage of cooperation, `https://www.researchers.one/article/2019-03-4`.