

Converse Jensen Inequality

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ABSTRACT

We use Skorokhod's embedding theorem to give a new proof of a converse to Jensen's inequality.

Let $(\Omega, \mathcal{F}, \mathbf{P})$ be a probability space and let X be an element of $L^1(\Omega, \mathcal{F}, \mathbf{P})$. Let $\mathcal{G} \subset \mathcal{F}$ be a σ -algebra, and define

$$Y := \mathbf{E}[X|\mathcal{G}],$$

an element of $L^1(\Omega, \mathcal{G}, \mathbf{P})$. If $\varphi : \mathbf{R} \rightarrow \mathbf{R}$ is convex then the conditional form of Jensen's inequality asserts that

$$\mathbf{E}[\varphi(X)|\mathcal{G}] \geq \varphi(\mathbf{E}[X|\mathcal{G}]) = \varphi(Y), \quad \text{a.s.},$$

part of the assertion being that the expectations are almost surely well-defined. In particular,

$$(1) \quad \mathbf{E}[\varphi(X)] \geq \mathbf{E}[\varphi(Y)],$$

with an analogous stipulation. The following converse assertion seems to be well known; see [2] and [3]. The proof we present may have some claim to novelty. We write $X \stackrel{d}{=} Y$ to indicate that random variables X and Y have the same distribution.

Theorem. *Let X and Y be integrable random variables such that (1) holds for all convex φ . Then there is a probability space $(\Omega', \mathcal{F}', \mathbf{P}')$, a random variable $X' \in L^1(\Omega', \mathcal{F}', \mathbf{P}')$, and a σ -algebra $\mathcal{G}' \subset \mathcal{F}'$ such that $X \stackrel{d}{=} X'$ and $Y \stackrel{d}{=} \mathbf{E}[X'|\mathcal{G}']$.*

Proof. Taking $\varphi(x) = x$ and then $\varphi(x) = -x$, we see that $\mathbf{E}[X] = \mathbf{E}[Y]$. Let $B = (B_t)$ be a one-dimensional Brownian motion defined on some filtered probability space $(\Omega', \mathcal{F}', \mathcal{F}'_t, \mathbf{P}')$, such that $B_0 \stackrel{d}{=} Y$. By Skorokhod's theorem, as presented in [1], condition (1) implies that there is an (\mathcal{F}'_t) -stopping time T such that $B_T \stackrel{d}{=} X$. But then $\mathbf{E}[B_T|\mathcal{F}'_0] = B_0$. Thus we take $X' = B_T$, and $\mathcal{G}' = \mathcal{F}'_0$. \square

References

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