On Cauchy's Functional Equation

by

P. J. Fitzsimmons

It is well known that the only Lebesgue measurable solutions $f\colon R \to R$ of the functional equation

$$f(x+y) = f(x) + f(y), \qquad \forall x, y \in R \tag{1}$$

are the linear functions $f(x) = f(1)x$. In fact, it is easy to see that any continuous solution of (1) is linear, and that any locally bounded solution is continuous. To show that any measurable solution of (1) is locally bounded, we can appeal to a result of Steinhaus: If A is a Lebesgue measurable subset of $[-1,1]$ with measure greater than $3/2$, then the set of differences of elements of A contains the entire interval $[-1,1]$. This fact, Egorov's theorem, and the additivity of f finish the job.

In this note we shall present a direct and elementary proof that (1) has only linear solutions in the class of Lebesgue measurable functions. No knowledge beyond the dominated convergence theorem will be needed.

Let f satisfy (1) and define a complex-valued function g by the rule $g(x) := \exp(if(x))$. Then (1) implies

$$g(x+y) = g(x)g(y). \tag{2}$$

The point of this transformation is that g is a bounded function, hence locally integrable. Since g never vanishes, we can find reals $a < b$ such that $\int_a^b g(t)\,dt \neq 0$. Integrating with respect to $y \in [a,b]$ in (2) we find that

$$\int_{a+x}^{b+x} g(t)\,dt = g(x)\left[\int_a^b g(y)\,dy\right]. \tag{3}$$

The left side of (3) is continuous in x by virtue of Lebesgue's dominated convergence theorem. Thus, g is a *continuous* solution of (2) with values in the unit circle of the complex plane. As such, there is a real ρ such that $g(x) = \exp(i\rho x)$ for all x. Consequently, there is an integer-valued function N such that

$$f(x) = \rho x + 2\pi N(x), \qquad \forall x. \tag{4}$$

Now notice that for each positive integer k, the preceding argument applies just as well to the function $x \mapsto k^{-1}f(x)$. This yields the existence of reals ρ_k and integer-valued functions N_k such that

$$f(x) = k\rho_k x + 2\pi k N_k(x), \qquad \forall x \in R, k = 2, 3, \ldots. \tag{5}$$

Subtracting (4) from (5) we see that $x(k\rho_k - \rho)/2\pi$ is an integer for all $x \in R$, so that $k\rho_k = \rho$. It follows that $N(x) = kN_k(x)$ for all $x \in R$ and all $k \geq 2$. The integer $N(x)$ therefore vanishes, being divisible by every positive integer. This means that $f(x) = \rho x$, as was to be shown.

Added Note (March 28, 2019) I learned from [**2**] of the paper [**1**] of Kac, which treats the Cauchy equation by the same argument presented here. Reem also poins out that the argument given in [**1**] and here requires only the Lebesgue measurability of $\exp(if)$.

References

[1] Kac, M.: Une remarque sur les équations fonctionnelles. *Comment. Math. Helv.* **9** (1937) 170–171.

[2] Reem, D.: Remarks on the Cauchy functional equation and variations of it. *Aequat. Math.* **91** (2017) 237–264.